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Ideal Elements for Perturbed Keplerian Motions *

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(January 30, 1975)

A new set of elements is proposed to describe Keplerian motions subject to perturbing forces. The resulting equations do not break down for small eccentricities, small inclinations and rotations of the ideal frame reference that are half turns. The parameters are selected with a view of simplifying the programming of the right-hand members.

Key words. Celestial mechanics; orbit perturbation; satellite ephemeris; variation of parameters.

1. Introduction

With the development of minicomputers and micropr cessors, the advantages of the method of variation of parameters (in short, *VOP*, also referred to as the variation of elements or the variation of constants), may outweigh, in many individual problems, the simplicity and generality of a straightforward numerical integration in Cartesian coordinates with respect to a fixed frame of reference (Cowell's method). These advantages include especially (1) a lesser accumulation of errors because the integration processes parameters that are affected solely by the usually small perturbations, (2) fewer steps of integration because the parameters are slowly varying, (3) a deeper insight in the physics of the problem because the elements are selected to enhance the special effects of the perturbing forces on the characteristics of a Keplerian motion.

There have been numerous attempts at finding the parameters best suited to the differential equations

$$\dot{x} = X, \tag{1}$$

$$\dot{X} = -\left(\frac{\mu}{r^3}\right) x + F \tag{2}$$

that describe the motion of a mass particle with position x and velocity X in a Newtonian potential of constant μ to which is superimposed a perturbing force F. Classical astronomers have been overly concerned that conversion from Cartesian coordinates to Keplerian elements not raise the order of the original system. For someone integrating numerically a differential system by consulting a table

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of logarithms or operating a desk calculator, higher order necessarily meant loss of efficiency. The breakthrough in adapting *VOP* to modern computing conditions was made by Samuel Herrick [1][1]. Abandoning the requirement that the order of the system be kept at its minimum, be developed into a computer algorithm the suggestion by Milankovič [2] that the Eulerian angles h, g, and I (respectively the right ascension of the node, the argument of the perigee and the inclination) be incorporated in the direction cosines of the major and minor axes of the reference ellipse. These unit vectors, henceforth denoted $\hat{a}$ and $\hat{b}$, when taken as parameters, produce an overabundant system, yet they render the right-hand members of the equations more symmetric, hence simpler to program and faster to evaluate. Applied to the orbit of Icarus, the new version of *VOP* amply justified Herrick's expectations [3].

Vectorial elements had been used by Stromgren [4], Bilimovič [5, 6], Musen [7], and Popovič [8] to derive equations for combinations of the classical Keplerian elements. Now following the lead taken by Herrick, Musen [9] proposed another computer algorithm in which the direction $\hat{b}$ of the minor axis and the normal to the orbital plane (henceforth denoted $\hat{n}$) are used as elements. Musen's algorithm has been the root of the computer programs developed at Cincinnati Observatory in relation with the Vanguard Project and the Minor Planets Center.

Substituting the couple of directions $(\hat{a}, \hat{b})$ or $(\hat{b}, \hat{n})$ for the Eulerian angles (h, I, g) amounts to referring perturbed Keplerian motions to a slowly moving frame. This viewpoint is not new; Hansen [10] designed his lunar theory around the concept of an *ideal* frame chosen to ensure a measure of independence to the motion in a plane from the rotation of the intermediate referential. Herget [11] seized on this idea to eliminate the singularity caused by small eccentricity in the *VOP* equations based on the apsidal frame $(\hat{a}, \hat{b}, \hat{n})$. But as he chose to represent the motion of the ideal frame as a finite rotation in order to keep the order of the system at its minimum, Herget introduces another kind of singularity, namely, those due to rotations that are half turns. We show here how Eulerian parameters eliminate the singularities due to small inclinations without generating spurious singularities at half turns.

Wider acceptance of the method of *VOP* is hampered by the lack of a systematic derivation of the various concepts and equations free from second-hand references to foreign journals and monographs out of circulation and in a mathematical language familiar to contemporary engineers. We have endeavored to restructure the piecemeal contributions brought to the theory of special perturbations over the last forty years. In the course of this work, we have come across a new set of elements that we think are particularly well suited to numerical integration by electronic computers.

2. Ideal Frames and Departure Points

The angular momentum (per unit of mass) of a particle is the vector

$$\mathbf{G} = \mathbf{x} \times \mathbf{X}; \tag{3}$$

let G and $\hat{n}$ be respectively its norm and its direction. Let $\hat{x}$ be the unit vector in the direction of $\underset{\sim}{x}$, and $\hat{y}$ the unit vector such that the triple $(\hat{x}, \hat{y}, \hat{n})$ makes an orthonormal base. We call it the *orbital frame*. By definition

$$\mathbf{G} = G\,\hat{n}, \tag{4}$$

$$\mathbf{x} = r\,\hat{x}, \tag{5}$$

$$\hat{y} = \hat{n} \times \hat{x}. \tag{6}$$

[1]Figures in brackets indicate the literature references at the end of this paper.

The base $(\hat{x}_0, \hat{y}_0, \hat{n}_0)$ that is the orbital frame at the epoch t_0 is adopted as the fixed frame of reference.

In the instantaneous orbital frame, the perturbing force is decomposed in the sum

$$\mathbf{F} = R\hat{x} + T\hat{y} + N\hat{n}; \tag{7}$$

the components R, T and N are sometimes referred to as the radial, the transversal and the normal components of the perturbation respectively.

Differentiation of (3) yields readily that

$$\dot{\mathbf{G}} = \mathbf{x} \times \mathbf{F}. \tag{8}$$

Then by differentiating the relation

$$G^2 = \mathbf{G} \cdot \mathbf{G},$$

one obtains that

$$G = rT, \tag{9}$$

and on considering (8) and (9) while differentiating (4), one gets finally that

$$\frac{d}{dt}\hat{n} = -\frac{r}{G}N\hat{y}. \tag{10}$$

Likewise, from the identity,

$$r^2 = \mathbf{x} \cdot \mathbf{x},$$

one deduces that

$$\dot{r} = \hat{x} \cdot \mathbf{X} \tag{11}$$

and comes, through (1), (2) and (11), to the basic equation

$$\frac{d}{dt}\hat{x} = \frac{G}{r^2}\hat{y}. \tag{12}$$

Finally differentiation of (6) via (10) and (12) produces the last equation

$$\frac{d}{dt}\hat{y} = -\frac{G}{r^2}\hat{x} + \frac{r}{G}N\hat{n} \tag{13}$$

necessary to describe the motion of the orbital frame.

Accordingly, the motion of the orbital frame with respect to the fixed frame may be viewed as the rotation

$$\frac{d}{dt}\hat{x} = \boldsymbol{\omega} \times \hat{x}, \frac{d}{dt}\hat{y} = \boldsymbol{\omega} \times \hat{y}, \frac{d}{dt}\hat{n} = \boldsymbol{\omega} \times \hat{n}$$

with the angular velocity

$$\boldsymbol{\omega} = \frac{G}{r^2}\hat{n} + \frac{r}{G}N\hat{x}. \tag{14}$$

So it turns out to be the product of

(1) a rotation from the fixed frame to an intermediate frame $(\hat{x}_I, \hat{y}_I, \hat{n})$ with the angular velocity

$$\omega_I = \frac{r}{G} N \hat{x}; \tag{15}$$

(2) a rotation from the intermediate frame $(\hat{x}_I, \hat{y}_I, \hat{n})$ to the orbital frame with the angular velocity

$$\omega_K = \frac{G}{r^2} \hat{n}. \tag{16}$$

In the absence of perturbations, $\omega = \omega_K$. This fact led Hansen to call the base $(\hat{x}_I, \hat{y}_I, \hat{n})$ an *ideal* frame. The direction $\hat{x}_I$ is called a *departure point* in the orbital plane.

Let θ be the angle from the departure point $\hat{x}_I$ to the position vector $\hat{x}$, so that

$$\hat{x} = \hat{x}_I \cos\theta + \hat{y}_I \sin\theta. \tag{17}$$

Then, according to definition (14), the attitude of the ideal frame with respect to the fixed frame is described by the differential system

$$\frac{d}{dt}\hat{x}_I = \omega_I \times \hat{x}_I = -\frac{r}{G} N \hat{n} \sin\theta, \tag{18}$$

$$\frac{d}{dt}\hat{y}_I = \omega_I \times \hat{y}_I = \frac{r}{G} N \hat{n} \cos\theta, \tag{19}$$

$$\frac{d}{dt}\hat{n} = \omega_I \times \hat{n} = \frac{r}{G} N(\hat{x}_I \sin\theta - \hat{y}_I \cos\theta). \tag{20}$$

The system is exempt of singularities for the class of Keplerian motions that are not linear (i.e., for which $G = 0$). But it is redundant since the unknowns $\hat{x}_I, \hat{y}_I, \hat{n}$ must satisfy at all times the invariant relations

$$\hat{x}_I \cdot \hat{x}_I = \hat{y}_I \cdot \hat{y}_I = \hat{n} \cdot \hat{n} = 1, \tag{21}$$

$$\hat{x}_I \cdot \hat{y}_I = \hat{y}_I \cdot \hat{n} = \hat{n} \cdot \hat{x}_I = 0. \tag{22}$$

Nevertheless, Andoyer [12] suggested to integrate numerically the first two equations while reserving the third one and the invariant relations to monitor the propagation of round off and truncation errors in the course of a numerical integration.

One way of eliminating the liaisons is to represent the rotation by its Eulerian angles. Let h be the longitude of the node of the instantaneous orbital plane in the orbital plane at epoch, I the inclination of the instaneous orbital plane over the orbital plane at epoch, and σ the argument of the departure point reckoned from the node in the instantaneous orbital plane. Note that $-\sigma$ is what classical authors call the *longitude of the departure point*. The following relations can be found in textbooks in kinematics:

$$\omega_I \cdot \hat{x}_I = \dot{h} \sin I \sin\sigma + \dot{I} \cos\sigma, \tag{23}$$

$$\boldsymbol{\omega}_I \cdot \hat{y}_I = \dot{h} \sin I \cos \sigma - \dot{I} \sin \sigma, \tag{24}$$

$$0 = \boldsymbol{\omega}_I \cdot \hat{n} = \dot{h} \cos I + \dot{\sigma}. \tag{25}$$

According to (25), once h and I have been determined, the argument σ results from a quadrature; in other words, it is determined only to the addition of an arbitrary constant. Therefore, one is free to select the departure point at epoch. The obvious choice is the direction $\underset{\circ}{\hat{x}}$ which makes the ideal frame at epoch coincide with the orbital frame at epoch.

Substituting definition (15) in the relations (23), (24) and (25) provides the differential equations in the Eulerian angles:

$$\dot{h} = \frac{r}{G} N \sin (\theta + \sigma) \operatorname{cosec} I, \tag{26}$$

$$\dot{I} = \frac{r}{G} N \cos (\theta + \sigma), \tag{27}$$

$$\dot{\sigma} = -\dot{h} \cos I. \tag{28}$$

The initial conditions are

$$h_0 = I_0 = \sigma_0 = 0, \tag{29}$$

which define precisely the point at which the system fails.

Various combinations of h and I have been proposed to eliminate the singularity $I = 0$ from the system (26), (27), (28). They derive in principle from the selection

$$k_1 = \sin I \cos h, \qquad k_2 = \sin I \sin h$$

made by de Sitter [13]. The resulting equations are usually given a form best suited to hand calculations by reference to a table of logarithms or by means of a desk calculator.

The instantaneous position of the ideal frame may be viewed as the image of the ideal frame at epoch by a *finite* rotation.

Rodriguez [14] represents a finite rotation by a vector $\mathbf{Q}$: its direction is the axis of rotation, its norm Q is set equal to $2 \tan \frac{1}{2}\chi$, where χ is the amplitude of the rotation.

If $\boldsymbol{\omega}$ is the angular velocity of the rotation, then

$$\dot{\mathbf{Q}} = \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega} \times Q + \frac{1}{4}(\mathbf{Q} \cdot \boldsymbol{\omega})\,\mathbf{Q}. \tag{30}$$

Applied to the rotation vector Q_I of the ideal frame, the basic identity (30) yields the differential vector equation

$$\dot{\mathbf{Q}}_I = \frac{r}{G} N \left[\hat{x} + \frac{1}{2}\hat{x} \times \mathbf{Q}_I + \frac{1}{4}(\mathbf{Q}_I \cdot \hat{x})\,\mathbf{Q}_I \right] \cdot \tag{31}$$

It is equivalent to either the differential system (18), (19), (20) or the scalar differential system (26), (27), (28). At epoch the finite rotation is the identity mapping, hence the set of initial conditions

$$\mathbf{Q}_I(t_0) = \mathbf{0}. \tag{32}$$

Equation (31) may be decomposed in the moving frame of reference. Since

$$\dot{\mathbf{Q}}_I = \frac{\partial}{\partial t}\,\mathbf{Q}_I + \boldsymbol{\omega}_I \times \mathbf{Q}_I$$

where, as usual, $\frac{\partial}{\partial t}$ denotes the time derivative in the moving frame, the vector eq (31) becomes

$$\frac{\partial}{\partial t}\,\mathbf{Q}_I = \frac{r}{G}\,N\left[\,\hat{x} - \frac{1}{2}\,\hat{x}\times\mathbf{Q}_I + \frac{1}{4}\,(\mathbf{Q}_I\cdot\hat{x})\,\mathbf{Q}_I\,\right]. \tag{33}$$

Whether, in the fixed frame or in the moving frame, the vector of finite rotation decomposes in the sum

$$\mathbf{Q}_I = Q_1\hat{x}_0 + Q_2\hat{y}_0 + Q_3\hat{n} = Q_1\hat{x}_I + Q_2\hat{y}_I + Q_3\hat{n},$$

and the vector eq (33) gives rise to the scalar system

$$\dot{Q}_1 = \frac{r}{G}\,N\left[\,\cos\theta - \frac{1}{2}\,Q_3\sin\theta + \frac{1}{4}\,(Q_1\cos\theta + Q_2\sin\theta)\,Q_1\,\right], \tag{34}$$

$$\dot{Q}_2 = \frac{r}{G}\,N\left[\,\sin\theta + \frac{1}{2}\,Q_3\cos\theta + \frac{1}{4}\,(Q_1\cos\theta + Q_2\sin\theta)\,Q_2\,\right], \tag{35}$$

$$\dot{Q}_3 = \frac{r}{G}\,N\left[\,-\frac{1}{2}\,(Q_2\cos\theta - Q_1\sin\theta) + \frac{1}{4}\,(Q_1\cos\theta + Q_2\sin\theta)\,Q_3\,\right]. \tag{36}$$

There results readily the invariant relation

$$\dot{Q}_1\,Q_2 - Q_1\,\dot{Q}_2 + \frac{1}{2}\,\dot{Q}_3 = 0 \tag{37}$$

to be used to monitor the integration of (31). It is strictly the correspondent of the quadrature (25).

Equation (31) is not used in *VOP*, for it can be simplified to the point where its programming becomes economical. Starting from Rodriguez identity

$$\hat{x} - \hat{x}_0 = \frac{1}{2}\,\mathbf{Q}_I \times (\hat{x} + \hat{x}_0), \tag{38}$$

one obtains in turn that

$$\hat{x} = \frac{1}{2}\,(\hat{x} + \hat{x}_0) + \frac{1}{4}\,\mathbf{Q}_I \times (\hat{x} + \hat{x}_0), \tag{39}$$

$$\hat{x}\times\mathbf{Q}_I = \frac{1}{2}\,(\hat{x} + \hat{x}_0)\times\mathbf{Q}_I + \frac{1}{4}\,Q_I^2\,(\hat{x} + \hat{x}_0) - \frac{1}{4}\,\mathbf{Q}_I\cdot(\hat{x} + \hat{x}_0)\,\mathbf{Q}_I, \tag{40}$$

$$\mathbf{Q}_I\cdot\hat{x} = \frac{1}{2}\,\mathbf{Q}_I\cdot(\hat{x} + \hat{x}_0). \tag{41}$$

On substituting (38), (40) and (41) into (30), one obtains eventually that

$$2\,\dot{\mathbf{Q}}_I = \frac{r}{G}\,N\,(1 + \frac{1}{4}\,\mathbf{Q}_I^2)\,(\hat{x} + \hat{x}_0), \tag{42}$$

which is one of the equations in Herget's *VOP*.

The differential equation for the vector of finite rotation is exempt of singularity at $I=0$. But, should the finite rotation approach a half turn, the amplitude χ would come close to π and the norm Q_I would grow beyond any finite bound. Equation (31) fails for half turns of the ideal frame. A sort of rectification should remedy this defect. Would the norm of $\mathbf{Q}_I$ becomes too large to be significant, the last obtained ideal frame should be taken as the fixed frame of reference. However easy in principle, this resetting of the epoch imposes various detailed adjustments in the programming of the *VOP*. We favor replacing the vectorial differential eq (39) by one that is definitely free from any sort of geometrical or kinematical singularities.

To this effect we choose to represent the vector of finite rotation Q_I by its Eulerian parameters, namely

$$\lambda_1 = (\mathbf{Q}_I \cdot \hat{x}_0) \sin \frac{1}{2}\chi = \sin \frac{1}{2} I \cdot \cos \frac{1}{2}(h-\sigma), \tag{43}$$

$$\lambda_2 = (\mathbf{Q}_I \cdot \hat{y}_0) \sin \frac{1}{2}\chi = \sin \frac{1}{2} I \cdot \sin \frac{1}{2}(h-\sigma), \tag{44}$$

$$\lambda_3 = (\mathbf{Q}_I \cdot \hat{n}) \sin \frac{1}{2}\chi = \cos \frac{1}{2} I \cdot \sin \frac{1}{2}(h+\sigma), \tag{45}$$

$$\lambda_0 \qquad \cos \frac{1}{2}\chi = \cos \frac{1}{2} I \cdot \cos \frac{1}{2}(h+\sigma). \tag{46}$$

As we remark that

$$\lambda_0^2 \left(1+\frac{1}{4}Q_I^2\right) = 1, \tag{47}$$

$$\mathbf{Q}_I \cdot \dot{\mathbf{Q}}_I = \left(1+\frac{1}{4}Q_I^2\right)(\mathbf{Q}_I \cdot \boldsymbol{\omega}_I), \tag{48}$$

we infer that

$$\dot{\lambda}_0 = \frac{1}{4}\lambda_0 (\mathbf{Q}_I \cdot \boldsymbol{\omega}), \tag{49}$$

$$\frac{d}{dt}\lambda_0 Q_I = \lambda_0 \left(\boldsymbol{\omega}_I + \frac{1}{2}\boldsymbol{\omega}_I \times \mathbf{Q}_I\right). \tag{50}$$

There remains now to replace $\lambda_0 \mathbf{Q}_I$ by its development

$$\lambda_0 \mathbf{Q}_I = 2(\lambda_1 \hat{x}_I + \lambda_2 \hat{y}_I + \lambda_3 \hat{n})$$

in (50) and it will turn out that the vector eq (33) is equivalent to the scalar system.

$$2\dot{\lambda}_1 = N(\lambda_0 u - \lambda_3 v), \tag{51}$$

$$2\dot{\lambda}_2 = N(\lambda_0 v + \lambda_3 u), \tag{52}$$

$$2\dot{\lambda}_3 = N(\lambda_1 v - \lambda_2 u), \tag{53}$$

$$2\dot{\lambda}_0 = N(-\lambda_1 u - \lambda_2 v). \tag{54}$$

We have just introduced the notations

$$u = \frac{r}{G} \cos \theta, \tag{55}$$

$$v = \frac{r}{G} \sin \theta, \tag{56}$$

to make the equations more concise and to underscore the places where the programming can be made efficient. The initial conditions are

$$\lambda_0 = 1, \qquad \lambda_1 = \lambda_2 = \lambda_3 = 0.$$

The equations are free from singularities, either geometric or kinematic. Moreover, the Eulerian parameters satisfy the inequalities

$$-1 \leqslant \lambda_i \leqslant +1, \qquad 0 \leqslant i \leqslant 3,$$

a circumstance that is especially favorable to a numerical integration. The counterpart of the quadrature (25) and of the invariant relation (37) is the liaison

$$\lambda_1 \dot{\lambda}_2 - \lambda_2 \dot{\lambda}_1 + \cdot \lambda_0 \lambda_3 - \dot{\lambda}_0 \lambda_3 = 0$$

which we readily infer from eq (51)–(54). In addition we have the identity

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$$

that is inherent to any quadruple of Eulerian parameters.

Eulerian parameters are somewhat unusual in celestial mechanics. They were considered first by Musen to describe the rotation of the apsidal frame [15]; the suggestion was then applied to the ideal frame [16] and followed by Broucke for the orbital frame [17].

3. The Osculating Ellipse in the Orbital Plane

Referred to the instantaneous ideal frame, the original equations of motion (1) and (2) become

$$\dot{\mathbf{x}} - \boldsymbol{\omega}_I \times \mathbf{x} = \frac{\partial}{\partial t} \mathbf{x} = \mathbf{X}, \tag{57}$$

$$\dot{\mathbf{X}} - \boldsymbol{\omega}_I \times \mathbf{X} = \frac{\partial}{\partial t} \mathbf{X} = -\frac{\mu}{r^3} \mathbf{x} + R\,\hat{x} + T\,\hat{y}. \tag{58}$$

Likewise the equation for the angular momentum converts into

$$\frac{\partial}{\partial t} \mathbf{G} = r\,T\,\hat{n}, \tag{59}$$

and the rotation of the orbital frame with respect to the instantaneous ideal frame is described by the equations

$$\frac{\partial}{\partial t} \hat{x} = \boldsymbol{\omega}_K \times \hat{x} = \frac{G}{r^2} \hat{y}, \tag{60}$$

$$\frac{\partial}{\partial t}\hat{y}=\omega_K\times\hat{y}=-\frac{G}{r^2}\hat{x}, \tag{61}$$

$$\frac{\partial}{\partial t}\hat{n}=\omega_K\times\hat{n}=0. \tag{62}$$

Whereas the concept of angular momentum is the root of our considerations about the rotation of the orbital plane, the Laplacian vector

$$\mathbf{A}=\mathbf{X}\times\mathbf{G}-\mu\hat{x} \tag{63}$$

is the cornerstone in our description of the osculating ellipse about the center of attraction in the orbital plane. The dominant role in the dynamics of the problem is played by the *effective* perturbing force

$$\mathbf{K}=\mathbf{F}-\frac{r}{p}\hat{x}\times(\hat{x}\times\mathbf{F}), \tag{64}$$

where

$$p=\frac{G^2}{\mu} \tag{65}$$

is the semiparameter of the ellipse. In the sequel the following identities

$$\mathbf{K}\cdot\hat{x}=F\cdot\hat{x},$$

$$\mathbf{K}\cdot\hat{y}=\left(1+\frac{r}{p}\right)\mathbf{F}\cdot\hat{y},$$

$$\mathbf{K}\cdot\hat{n}=\left(1+\frac{r}{p}\right)\mathbf{F}\cdot\hat{n}$$

will be used repeatedly without being referred to specifically.

Cross multiplication of (63) by $\hat{n}$ yields the equation of the hodograph

$$\mathbf{X}=\frac{1}{G}\hat{n}\times\mathbf{A}+\frac{\mu}{G}\hat{y} \tag{66}$$

and brings us to consider the vector

$$\mathbf{B}=\frac{1}{G}\hat{n}\times\mathbf{A} \tag{67}$$

in the direction of the minor axis.

On introducing definition (67) into the equation of the hodograph (66) and then differentiating with respect to time in the instantaneous ideal frame as per eq (58) and (61), one finds eventually that

$$\frac{\partial}{\partial t}\mathbf{B}=\mathbf{K}-(\mathbf{K}\cdot\hat{n})\hat{n}=(\hat{n}\times\mathbf{K})\times\hat{n}. \tag{68}$$

Let $\hat{b}$ be the direction of **B**; also let μe denote the norm of the Laplacian vector. Then (67) takes the form

$$\mathbf{B}=\frac{\mu e}{G}\hat{b} \tag{69}$$

and, on using (68), one finds readily that

$$\frac{d}{dt}\frac{\mu e}{G} = \mathbf{K} \cdot \hat{b}, \tag{70}$$

which is the most concise expression we could find for the time derivative of the eccentricity.

If $\hat{a}$ denotes the direction of A, then $\hat{n} = \hat{a} \times \hat{b}$ and the triple $(\hat{a}, \hat{b}, \hat{n})$ constitutes that orthonormal base that we proposed in the introduction to call the *apsidal frame*. With the help of (68), (69), and (70), one checks readily that

$$\boldsymbol{\omega}_A = -\frac{G}{\mu e}\,(\mathbf{K} \cdot \hat{a})\,\hat{n} \tag{71}$$

is the angular velocity of the apsidal frame with respect to the instantaneous ideal frame, so that the rotation of the apsidal frame is described by the equations

$$\frac{\partial}{\partial t}\,\hat{a} = \boldsymbol{\omega}_A \times \hat{a} = -\frac{G}{\mu e}\,(\mathbf{K} \cdot \hat{a})\,\hat{b}, \tag{72}$$

$$\frac{\partial}{\partial t}\,\hat{b} = \boldsymbol{\omega}_A \times \hat{b} = \frac{G}{\mu e}\,(\mathbf{K} \cdot \hat{a})\,\hat{a}, \tag{73}$$

$$\frac{\partial}{\partial t}\,\hat{n} = \boldsymbol{\omega}_A \times \hat{n} = 0.$$

The argument of perigee g is defined here as the angle from the departure point $\hat{x}_I$ to the semi-major axis $\hat{a}$. In order to circumvent the indetermination due to small eccentricities, we consider the functions

$$C = \frac{\mu e}{G}\cos g, \tag{74}$$

$$S = \frac{\mu e}{G}\sin g. \tag{75}$$

The equations for C and S result from differentiating the sum

$$\frac{\mu e}{G}\,\hat{a} = C\,\hat{x}_I + S\,\hat{y}_I$$

in the ideal frame to obtain that

$$\dot{C}\,\hat{x}_I + \dot{S}\,\hat{y}_I = \frac{d}{dt}\left(\frac{\mu e}{G}\right) \cdot \hat{a} + \frac{\mu e}{G}\frac{\partial}{\partial t}\,\hat{a} = K \times \hat{n},$$

thus leading to the simple (and probably original) equations

$$\dot{C} = K \cdot \hat{y}_I, \tag{76}$$

$$\dot{S} = -K \cdot \hat{x}_I. \tag{77}$$

The motion of the osculating ellipse in the orbital plane with respect to the instantaneous ideal frame is completely determined by the system consisting of eq (9), (76) and (77).

Special care must be taken while computing a position on the osculating ellipse in order not to bring in singularities (e.g., divisions by e) or indeterminations (e.g., appearances of sin g and cos g in other combinations than C and S) caused by small eccentricities. We postpone until the next section dealing with the problem of timing the motion along the osculating ellipse, and so assume that the mean longitude

$$F = l + g \tag{78}$$

reckoned from the departure point is known. Once C and S have been obtained, we revert to the more commonly used elements

$$X = \frac{G}{\mu} C = e \cos g, \tag{79}$$

$$Y = \frac{G}{\mu} S = e \sin g. \tag{80}$$

We also consider the eccentric anomaly

$$\phi = E + g \tag{81}$$

reckoned from the departure point. The classical Kepler equation

$$l = E - e \sin E$$

may be extended into an implicit equation relating ϕ to F:

$$F = \phi - (X \sin \phi - Y \cos \phi). \tag{82}$$

Likewise we transform the classical identities related to the anomalies reckoned from the line of apsides $\hat{a}$

$$r = a\ (1 - e \cos E),$$

$$r \cos f = a\ (\cos E - e),$$

$$r \sin f = a\ \eta \sin E,$$

in which

$$\eta = (1 - e^2)^{1/2},$$

into formulas involving θ and ϕ reckoned from the point of departure:

$$\frac{r}{a} = 1 - X \cos \phi + Y \sin \phi, \tag{83}$$

$$\frac{r}{a} \cos \theta = \cos \phi - X + \frac{\phi - F}{1 + \eta} Y, \tag{84}$$

$$\frac{r}{a}\sin\theta = \sin\phi - Y - \frac{\phi - F}{1+\eta}X, \tag{85}$$

with

$$\eta = (1 - X^2 - Y^2)^{1/2}; \tag{86}$$

$$a = \frac{G^2}{\mu\eta^2} \tag{87}$$

4. The Timing on the Osculating Ellipse

Most authors borrow the differential equation for F from Brown [18]. His derivation rests on the use of a certain differential operator which he denotes δ; the present exposition shows that it is not needed. Also Brown's construction dates from a time when astronomers were not yet aware of vector calculus.

We start with writing the time derivative of the mean anomaly as follows:

$$\dot{l} = \dot{E} - \frac{d}{dt}(e\sin E); \tag{88}$$

each term in the right-hand member will be calculated separately.

Introducing the equation of the hodograph (66) into (11) results in the expression

$$\dot{r} = \frac{\mu e}{G}\sin f \tag{89}$$

Now, by definition of the eccentric anomaly,

$$r\sin f = a\,\eta\sin E;$$

hence, formula (89) takes the form

$$r\,\dot{r} = L\,e\sin E \tag{90}$$

provided that

$$L = (\mu\,a)^{1/2} \tag{91}$$

denotes Delaunay's action variable. Actually (90) yields the time derivative of $e\sin E$ as the combination

$$\frac{d}{dt}e\sin E = \frac{1}{L}\left[\frac{d}{dt}(r\,\dot{r}) - r\,\dot{r}\,\frac{d}{dt}\log L\right].$$

We evaluate

$$\frac{d}{dt}r\,\dot{r} = \frac{d}{dt}(\mathbf{x}\cdot\mathbf{X}) = (\mathbf{X}\cdot\mathbf{X}) + \left(\mathbf{x}\cdot\frac{d}{dt}X\right)$$

$$= (\mathbf{X}\cdot\mathbf{X}) - \frac{\mu}{r} + (\mathbf{x}\cdot F) = -\frac{\mu}{a} + \frac{\mu}{r} + (\mathbf{x}\cdot F)$$

and obtain eventually that

$$\frac{d}{dt}e\sin E = -n + \frac{r}{L}(\mathbf{K}\cdot\dot{\mathbf{x}}) + \left[n\frac{a}{r} - \frac{r\,\dot{r}}{L}\frac{d}{dt}\log L\right]. \tag{92}$$

In order to compute $\dot{E}$, we start from the identity

$$\frac{\dot{E}}{\sin E} = \frac{d}{dt} \log \tan \frac{E}{2}$$

which, in view of the classical formula

$$\tan \frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{f}{2},$$

transposes into the identity

$$\frac{\dot{E}}{\sin E} = -\frac{\dot{e}}{\eta^2} + \frac{\dot{f}}{\sin f}. \tag{93}$$

Reduction of the first term in the latter sum goes as follows

$$\frac{\dot{e}}{\eta^2} = \frac{e^2+\eta^2}{\eta^2}\,\dot{e} = e\left(\frac{d}{dt}\log e - \frac{d}{dt}\log \eta\right)$$

$$= e\left(\frac{d}{dt}\log \frac{\mu e}{G} + \frac{d}{dt}\log L\right).$$

On account of (70), the reduction may proceed further to

$$\frac{\dot{e}}{\eta^2} = \frac{G}{\mu}\,(\mathbf{K}\cdot\dot{b}) + e\,\frac{d}{dt}\log L. \tag{94}$$

As for the second term in the right-hand member of (93), we observe that

$$\dot{f} = \dot{\theta} - \dot{g}, \tag{95}$$

$$\dot{\theta} = \frac{G}{r^2}, \tag{96}$$

$$\dot{g} = -\frac{G}{\mu e}\,\mathbf{K}\cdot\hat{a} = \boldsymbol{\omega}_A \cdot \hat{n}. \tag{97}$$

Equation (96) is a consequence of the hodograph eq (66). Indeed from taking the time derivative of x in the instantaneous frame of reference, we obtain immediately that

$$\mathbf{X} = \frac{\partial}{\partial t}\,\mathbf{x} = \dot{r}\,\hat{x} + r\,\dot{\theta}\,\hat{y},$$

so that

$$r\,\dot{\theta} = \mathbf{X}\cdot\hat{y} = \frac{\mu}{G}\,(1+e\cos f) = \frac{\mu p}{Gr} = \frac{G}{r}.$$

As, for eq (97), it derives readily from differentiating the definition (74) on account of (76).

By combining expressions (94)–(97), we expand (93) into the sum

$$\dot{E}=\frac{Gr}{\mu a\eta e}\,\mathbf{K}\cdot(\hat{a}-\hat{b}e\sin f)+\left[n\,\frac{a}{r}-\frac{r\dot{r}}{L}\,\frac{d}{dt}\log L\right]\cdot \tag{98}$$

Before we combine (92) and (98) we observe that

$$\hat{a}-\hat{b}e\sin f=\frac{p}{r}\,\hat{a}-e\hat{x}.$$

Eventually we reach a rather concise equation for the mean anomaly:

$$\dot{l}=n+\frac{G\eta}{\mu e}\,(\mathbf{K}\cdot\hat{a})-2\,\frac{r}{L}\,(\mathbf{K}\cdot\hat{x}). \tag{99}$$

As expected, it is singular at $e=0$. The critical point is eliminated by adding (97) and (99), which produce the equation for the mean longitude

$$\dot{F}=n+\frac{Ge}{\mu(1+\eta)}\,(\mathbf{K}\cdot\hat{a})-2\,\frac{r}{L}\,(\mathbf{K}\cdot\hat{x}). \tag{100}$$

Eventually to avoid having to determine the vector $\hat{a}$, we decompose the right-hand member of (100) in the instantaneous ideal of frame, which produces an equation exempt of any problems for small eccentricities:

$$\dot{F}=n+\frac{p}{\mu(1+\eta)}\,(C\dot{S}-\dot{C}S)+2\eta(u\dot{S}-v\dot{C}). \tag{101}$$

5. Conclusions

Completing the project opened by Herrick, developed by Musen and revised by Herget, table 1 summarizes the final version of a *VOP* free of geometrical and kinematical singularities and indeterminacies, with the right-hand members prepared for as efficient a coding as possible.

The elements $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, G, C, S, F)$ are termed ideal because they refer to an ideal frame of reference.

The derivation presented in this communication departs from the general methods of the theory of perturbation based on the Lagrange brackets for the conventional elliptical elements [19]. Rather it is based on the method of variation applied immediately to the vector integrals (angular momentum and Laplacian vector) of the problem of two bodies and the well-known formulas for differentiation in moving frames of reference. We concur with Lure [20] and Danby [21] in thinking that this is the most simple, informative and economical of all extant derivations.

TABLE 1. *Variations of parameters*

Attitude of the ideal frame	*Ellipse in the orbital plane*
$2\dot{\lambda}_1=N(\lambda_0 u-\lambda_3 v)$	$\dot{G}=rT$
$2\dot{\lambda}_2=N(\lambda_0 v+\lambda_3 u)$	$\dot{C}=K\cdot\hat{y}_I$
$2\dot{\lambda}_3=N(\lambda_1 v-\lambda_2 u)$	$-\dot{S}=K\cdot\hat{x}_I$
$-2\dot{\lambda}_0=N(\lambda_1 u+\lambda_2 v)$	

Timing

$$\dot{F} = n + \frac{p}{\mu(1+\eta)}(C\dot{S} - \dot{C}S) + 2\eta(u\dot{S} - v\dot{C})$$

Invariant relations

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$$
$$\lambda_0\dot{\lambda}_3 - \dot{\lambda}_0\lambda_3 + \lambda_1\dot{\lambda}_2 - \dot{\lambda}_2\lambda_1 = 0$$

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Generalized or Adjoint Reciprocity Relations for Electroacoustic Transducers*

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The "equations of motion" of a linear electroacoustic transducer are written in the alternative forms $L^+X=0$, $L^-X=0$, where the matrices $L^\pm$ are linear differential expressions, $X=(\bar{E}, \bar{H}, \bar{u})$ represents electroacoustic fields, and the superscripts distinguish selected normalizations of the equations of motion. To each operator $L^\pm$ corresponds a mathematically defined adjoint operator $(L^\pm)^a$ and an associated adjoint transducer, whose material tensor parameters are given by certain transpositions and interchanges of the parameters of the given transducer. Dissipative characteristics (lossiness, losslessness, or "gaininess") of the material of the given transducer are preserved pointwise in the adjoint transducers. A generalized reciprocity lemma leads to relations of reciprocal type between external properties of the given and the adjoint transducers. In the self-adjoint cases, the conventional electroacoustic reciprocity and antireciprocity relations are obtained and the derivation of those relations is critically confirmed. The generalized or adjoint reciprocity relations have been applied in the plane-wave scattering-matrix formalism developed for electroacoustic transducers by D. M. Kerns.

Corollaries of the adjoint reciprocity relations, analogous to conventional reciprocity theorems, but involving properties of adjoint pairs of transducers, are derived. Examples are discussed of transducers for which it is feasible to form the adjoint transducers.

Key words: Linear differential operators; reciprocity relations; scattering matrix; transducers.

Glossary of Symbols

a-superscript:	Denotes the adjoint.
t-subscript:	Denotes the transpose.
A_0, A_p, A:	Surface of transducer; surface of power supply or detector; and surface of transducer-plus-(power supply or detector).
a_0, b_0:	Amplitude of incident and emergent em mode which feeds the transducer.
$a(\bar{K}), b(\bar{K})$:	Incoming and outgoing plane-wave amplitudes in ambient fluid.
$\bar{\bar{\alpha}}, \bar{\bar{\alpha}}'$:	Direct and converse piezoelectric stress tensor (triadic).
$\mathscr{A}$:	Amplitude of spherical wave in ambient fluid.
$\bar{B}, \bar{B}_0$:	Harmonic and static magnetic induction.
$\bar{\bar{\beta}}, \bar{\bar{\beta}}'$:	Direct and converse piezomagnetic stress tensor (triadic).
$\bar{\bar{\bar{c}}}$:	Hooke's tensor (tetradic).
$\bar{D}$:	Electric displacement.
$\bar{E}, \bar{E}_0$:	Harmonic and static electric field.
$\bar{e}_0, \bar{h}_0$:	Electric and magnetic field components of em mode which feeds the transducer.

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$\bar{\bar{\epsilon}}$: Permittivity dyadic.
$\eta = \gamma/\omega\rho_0$: Plane-wave admittance.
η_0: Admittance of em mode feeding transducer.
$\bar{F}$: Bilinear concomitant associated with the operator L.
$\bar{f}$, $\bar{f}_s$: Induced and applied volume forces.
γ: z-component of the plane-wave propagation vector.
$\bar{H}$: Magnetic field.
h, h': Spacial impedances describing transducer.
$\bar{\bar{I}}$: Identity dyadic.
$\bar{J}$, $\bar{J}_s$: Induced and applied current densities.
$\bar{K}$, $\bar{L}$: Transverse part of $\bar{k}$ and $\bar{l}$.
$\bar{k}$, $\bar{l}$: Plane-wave propagation vectors in ambient fluid.
L, L^-: Transducer operator matrices.
$\bar{M}_s$: Applied magnetization.
$M(\bar{r})$: Microphone response.
$\bar{\bar{\mu}}$: Permeability dyadic.
μ_0: Free-space permeability.
p, p_0: Harmonic and static pressure.
$\bar{P}_s$: Applied polarization.
ρ_0: Static mass density of the ambient fluid.
ρ_m, ρ_m°, ρ_m': Harmonic, static, and total mass density of transducer material.
ρ_e, ρ_e°: Harmonic and static charge density.
S_0: Transducer feed surface.
S_1: Reference xy plane in ambient fluid.
S_{00}, S_{01}, S_{10}, S_{11}: Plane-wave scattering matrices describing transducer.
$S(\bar{r})$: Speaker response.
$\bar{\bar{s}}$: Strain dyadic.
$\bar{\bar{\sigma}}$: Conductivity dyadic.
$\bar{\bar{T}}$, $\bar{\bar{T}}_0$, $\bar{\bar{T}}'$: Harmonic, static, and total stress dyadic.
$\bar{\bar{\tau}}$, $\bar{\bar{\nu}}$: Magneto-electric dyadics.
$\bar{u}$: Fluid or material velocity.
V_0, I_0: Input voltage and current to transducer.
$V(\bar{K})$, $I(\bar{K})$: Voltage and current spectral densities.
Z_b, Z_0: Spacial impedances (along with h, h') describing transducer.

1. Introduction

Analytical techniques developed by Kerns [1, 2, 3][1] for the measurement of microwave antennas have been "translated" recently by that author into corresponding techniques for the measurement of electroacoustic transducers in fluids. The basic theory is formulated with a scattering-matrix description and emphasizes the use of plane-wave spectra for the representation of acoustic fields. For a summary and for details of the theory and its variety of applications, reference should be made to Kerns [4].

Although the electromagnetic (em) theory is quite comprehensive, the acoustics applications are strengthened by generalized or adjoint reciprocity relations [5], the derivation of which forms the principal subject of the present paper. The generalized reciprocity relations are expressed in the notation of the Kerns plane-wave scattering-matrix (*PWSM*) description, and also in the notation of the spacial impedance-matrix (*SIM*) description introduced by Foldy and Primakoff [6, 7] in their classic papers on electroacoustic transducers and reciprocity. The *PWSM* expres-

[1]Figures in brackets indicate the literature references at the end of this paper.

sions of adjoint reciprocity have become especially important because of their use in the theoretical foundation of extrapolation techniques which predict gain on an arbitrary axis from measurements at reduced distances along that axis [4]. (Extrapolation techniques were introduced for antennas by Wacker and Bowman [8].)

The following approach is taken to establish the adjoint reciprocity relationships. The equations of motion and constitutive relations describing a large class of linear electroacoustic transducers are reduced to the form of a linear operator matrix. The operator is "transposed" to form a unique adjoint operator, which is compared and combined with the original operator to derive a generalized electroacoustic reciprocity lemma relating the fields of the given transducer to those of an adjoint transducer. The material parameters of the adjoint transducer are determined by certain transpositions and interchanges of the material parameters of the given transducer. The reciprocity lemma leads to the generalized reciprocal relations between the characteristic (*PWSM* and *SIM*) matrices of the given and adjoint transducers. In addition, corollaries of the adjoint reciprocity relations analogous to conventional reciprocity theorems are derived for adjoint pairs of transducers.

Power relations and the associated "value classes" are investigated also for electroacoustic transducers and their adjoints. In particular, it is shown that the dissipative characteristics (lossiness, losslessness, or "gaininess") of the given transducer are preserved pointwise in the adjoint transducers.

Although a unique adjoint corresponds to a given operator, many different operators can represent the equations of the same transducer. For example, multiplication of one of the equations of motion by a constant will change the operator matrix without changing the content of the equation of motion. Analysis shows, however, that for a given transducer only two operators, L^+ and L^- (from here on the "+" will be omitted from L^+), have an associated adjoint operator, L^a and L^{-a}, which represents the equations of another transducer (the adjoint transducer). A transducer self-adjoint ($L^a = L$ or $L^{-a} = L^-$) with respect to L or L^- corresponds to the conventional reciprocal or antireciprocal transducer respectively. One of the advantages of the linear operator approach is that it facilitates the identification of self-adjoint transducers.

A number of experimental measurement techniques applicable to reciprocal (or antireciprocal) transducers only, could be extended to nonreciprocal transducers if the adjoint transducers were producible in the laboratory. In general, the difficulties in synthesizing the necessary electroacoustic material parameters may discourage or prevent the creation of the adjoint transducers. However, two groups of transducers are discussed for which the creation of an adjoint transducer involves, in principle, a simple modification of the given transducer.

2. The Transducer System

The electroacoustic transducer system under consideration is pictured in figure 1. The transducer is bounded by the closed, finite surface A_0. The em power supply or detector, which is bounded by the closed, finite surface A_p, feeds the transducer through the area S_0 common to both A_0 and A_p. The transducers, which may contain static bias fields and their sources, are termed "passive" if they cannot radiate more power than they absorb. The present analysis, however, is not limited to passive transducers, but may include transducers which are active because they contain "gainy" material, or radiating sources, or both. (A transducer will be called source-free if its internal equations of motion are homogeneous, i.e., if they are satisfied by all fields equal to zero.)

The transducer-plus-(power supply or detector) is immersed in a homogeneous, isotropic, stationary fluid which extends to infinity and supports the time harmonic ($e^{-i\omega t}$, $\omega > 0$) acoustic pressure-velocity field of small (first order differential) amplitude. The static pressure, static density, and propagation constant (the latter two of which may be complex function of real frequency ω

to account for viscous and "expansive friction" losses) are constants in space and time denoted by P_0, ρ_0 and k respectively. The fluid between the infinite xy plane $S_1(z=0)$ and the external surface

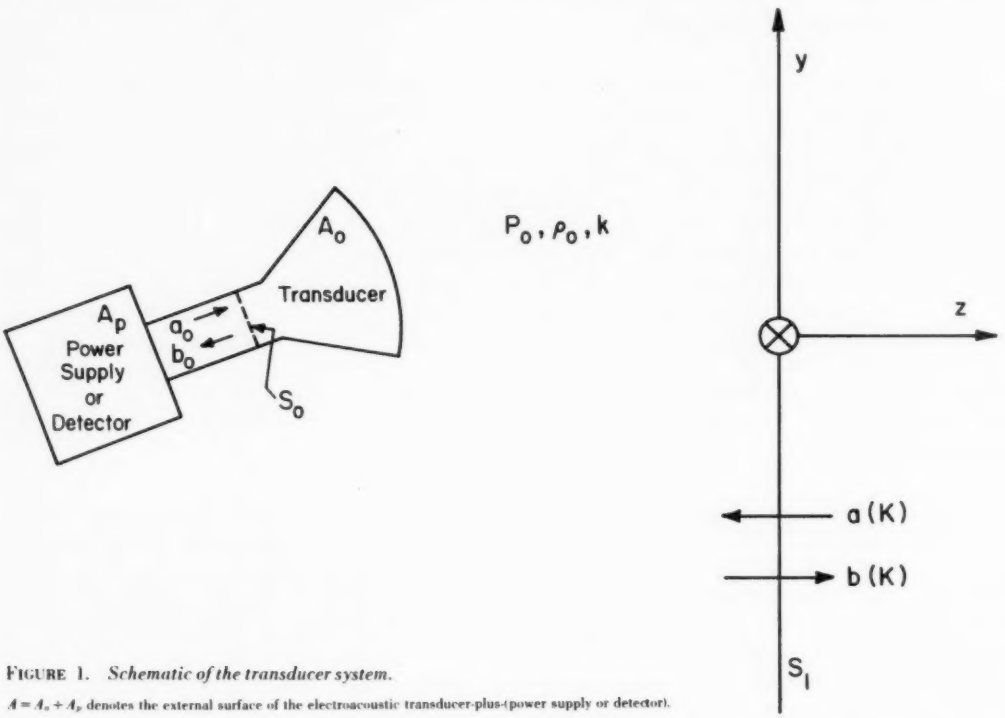

FIGURE 1. *Schematic of the transducer system.*

$A = A_0 + A_p$ denotes the external surface of the electroacoustic transducer-plus-(power supply or detector).

A of the transducer-plus-(power supply or detector) contains no acoustic sources,[2] although arbitrary sources (at frequency ω) may exist to the right of S_1.

The normal velocity on the surface A_p of the power supply-detector, including the feed area S_0, is assumed negligible. Electromagnetic shielding necessary for deriving the adjoint reciprocity relations is discussed in section 5.

The area S_0 may designate the perpendicular cross section of an open or closed em waveguide (uniform and isotropic) in which a single em mode is propagating with incident amplitude a_0 and emergent amplitude b_0, or simply the cross section of two wire leads when the frequency ω is low enough to allow analysis by quasistatic voltages and currents. The cross-sectional area S_0 for a mode on an open waveguide (such as a Lecher line or microstrip) must extend sufficiently far from the guide to insure that the fields in the mode outside S_0 contribute negligibly to the normalization integral ((7) below). The surfaces A_p, S_0, and A_0 represent imaginary boundaries in that they are chosen to facilitate the theoretical analysis and need not coincide with the physical boundaries of the system. Of course, the surface $A = A_0 + A_p$ must be chosen within (or on the interior boundary of) the ambient fluid.

The continuity and momentum equations, which govern the harmonic acoustic fields in the source-free fluid between the surface A and the infinite plane S_1, may be expressed as

$$k^2 p = -i\omega\rho_0 \nabla \cdot \bar{u} \tag{1a}$$

$$\nabla p = i\omega\rho_0 \bar{u}, \tag{1b}$$

[2] Between A and S_1 there can be finite regions of either acoustically rigid material or material in which volume forces are zero and stress is linearly related to strain by a symmetric Hooke's tensor. However, the reference plane S_1 must remain entirely in the homogeneous ambient fluid.

where the complex amplitudes, p and $\bar{u}$, denote the position-dependent excess pressure and fluid velocity. The effect of gravity on harmonic variation in mass density is neglected.

The excess pressure and normal velocity on the surface S_1 may be expanded in a double Fourier integral (transform) over plane wave voltage and current spectral densities [4]:

$$p(\bar{R}) = \frac{1}{2\pi} \int_{\bar{K}} V(\bar{K}) e^{i\bar{K}\cdot\bar{R}} \, d\bar{K} \tag{2a}$$

$$u_z(\bar{R}) = -\frac{1}{2\pi} \int_{\bar{K}} I(\bar{K}) e^{i\bar{K}\cdot\bar{R}} \, d\bar{K}. \tag{2b}$$

Conversely, the plane-wave voltage and current densities, which are defined in terms of the incoming $a(\bar{K})$ and outgoing $b(\bar{K})$ acoustic plane-wave amplitudes,

$$V(\bar{K}) = a(\bar{K}) + b(\bar{K}) \tag{3a}$$

$$I(\bar{K}) = [a(\bar{K}) - b(\bar{K})]\eta(K), \tag{3b}$$

may be expanded in a double Fourier integral (inverse transform) over the excess pressure and normal velocity:

$$V(\bar{K}) = \frac{1}{2\pi} \int_{S_1} p(\bar{R}) e^{-i\bar{K}\cdot\bar{R}} \, d\bar{R} \tag{4a}$$

$$I(\bar{K}) = -\frac{1}{2\pi} \int_{S_1} u_z(\bar{R}) e^{-i\bar{K}\cdot\bar{R}} \, d\bar{R}. \tag{4b}$$

The vector $\bar{K} = K_x\hat{i} + K_y\hat{j}$ can be interpreted as the transverse part of the plane-wave propagation vector $\bar{k}$. The vector R is confined to the xy plane S_1, and $\eta(K)$ equals $\gamma/\omega\rho_0$, where γ is defined as $(k^2 - K^2)^{1/2}$ with the sign of the radical chosen to keep the real and imaginary parts of γ positive.

The transverse em fields $(\bar{E}_t, \bar{H}_t)$ on the cross section S_0 of a waveguide may be written as the sum of the transverse electric $\bar{e}_0$ and magnetic $\bar{h}_0$ components of the mode traveling to the right with incident amplitude a_0 and to the left with emergent amplitude b_0:

$$\bar{E}_t = V_0\bar{e}_0, \qquad \bar{H}_t = I_0\bar{h}_0, \tag{5}$$

where

$$V_0 = a_0 + b_0 \tag{6a}$$

$$I_0 = (a_0 - b_0)\eta_0. \tag{6b}$$

If the dimensions of $\bar{e}_0$ and $\bar{h}_0$ are chosen as (meter)$^{-1}$ and $(\bar{e}_0, \eta_0\bar{h}_0)$ are consistent with Maxwell's equations, V_0 behaves dimensionally as volts, I_0 as amperes, and η_0 as an admittance (ohm^{-1}) normally chosen as a positive real number. Moreover, normalization may be expressed as a nondimensional number equal to unity, i.e.,

$$\int_{S_0} (\bar{e}_0 \times \bar{h}_0) \cdot \hat{n} \, da = 1. \tag{7}$$

where $\hat{n}$ is the inward normal to the transducer. (The rationalized MKS system of units is used throughout.) If the area S_0 simply cuts two wire leads at quasi-static frequencies, V_0 and I_0 refer

to conventional circuit voltages and currents which do not necessarily serve as genuine modal coefficients. In that case, (6) become a definition of a_0 and b_0.

The *PWSM* description relates the outgoing amplitudes, b_0 and $b(\bar{K})$, to the incoming amplitudes, a_0 and $a(\bar{K})$, by the scattering equations (see ref. [4]),

$$b_0 = S_{00}a_0 + \int_L S_{01}(\bar{L})\,a(\bar{L})\,d\bar{L} \tag{8a}$$

$$b(\bar{K}) = S_{10}(\bar{K})a_0 + \int_L S_{11}(\bar{K},\bar{L})a(\bar{L})\,d\bar{L} \tag{8b}$$

The scattering matrix is resolved into the four submatrices S_{01}, S_{10}, S_{11} and S_{00} which represent the receiving, radiating, acoustic scattering, and em reflection properties respectively for the electroacoustic transducer. Like $\bar{K}$, the vector $\bar{L}$ can be interpreted as the transverse part of the propagation vector $\bar{l}$.

Alternatively, the *SIM* relations[3] characterize the external behavior of the transducer by relating the "voltage" V_0 and excess pressure p on the surface of the transducer A_0 (except possibly on the feed area S_0) to the "current" I_0 and the inward (to transducer) normal velocity u_n on the surface A_0:

$$V_0 = Z_b I_0 + \int_{A_0} h'(\bar{r}_0)u_n(\bar{r}_0)\,da_0 \tag{9a}$$

$$p(\bar{r}) = h(\bar{r})I_0 + \int_{A_0} Z_0(\bar{r},\bar{r}_0)u_n(\bar{r}_0)\,da_0. \tag{9b}$$

Although three of the four spatial impedances (Z_b, h', h, Z_0) depend upon the position vectors $(\bar{r},\bar{r}_0)$ to the surface A_0 and all four depend upon the harmonic time frequency, they are assumed independent of the medium or sources surrounding the transducer and, as linearity implies, independent of V_0, I_0, p and u_n.

It can be shown [9], without reference to the internal behavior of the transducers, that the *PWSM* equations (8) imply the *SIM* relations (9) but not vice versa. That is, the *PWSM* description applies to a larger class of transducers than do the *SIM* relations. However, the alternate expressions of reciprocity (and antireciprocity),

$$\eta_0 S_{01}(\bar{K}) = \pm\,\eta(K)S_{10}(-\bar{K}) \tag{10a}$$

$$\eta(K)S_{11}(\bar{K},\bar{L}) = \eta(L)S_{11}(-\bar{L},\bar{K}), \tag{10b}$$

and

$$h'(\bar{r}) = \pm\,h(\bar{r}) \tag{11a}$$

$$Z_0(\bar{r},\bar{r}_0) = Z_0(\bar{r}_0,\bar{r})\ [6, 7], \tag{11b}$$

which are derived in section 5, can be proven [9] equivalent for transducers which obey both descriptions.

In analogy to the 2-port cases discussed by McMillan [10], Kerns [4] defines reciprocity, antireciprocity, and nonreciprocity for electroacoustic transducers in terms of whether (10a) holds with the plus sign, the minus sign, or not at all. Foldy and Primakoff [6,7] do not use the term "antireciprocity" explicitly, but distinguish between the plus and minus sign in (11a) by con-

[3] Foldy and Primakoff refer to equations (9) as the "linearity relations." Here they are referred to as the "spacial impedance-matrix relations" to avoid confusion with the spectral scattering-matrix equations (8), which are also linear.

sidering reciprocity through electric-type coupling and magnetic-type coupling.

Equations which characterize the internal behavior of the electroacoustic transducers under consideration are introduced in the following section.

3. The Operator Matrix Description

Primakoff and Foldy [7], in their original work on electroacoustic reciprocity, examined the equations which model the internal operation of a large class of linear transducers. They found that if certain "symmetry conditions" were satisfied by its material parameters and static fields, the transducers obeyed the *SIM* reciprocity relations (11). The present derivation begins in a similar way by writing the linear, harmonic ($e^{-i\omega t}$) equations that govern the "physics" of the material within the transducer A_0:

$$\nabla \cdot \bar{\bar{T}} + i\omega\rho_m^\circ \bar{u} + \bar{f} = 0 \tag{12}$$

$$\nabla \times \bar{E} - i\omega\bar{B} = 0 \tag{13a}$$

$$\nabla \times \bar{H} + i\omega\bar{D} - \bar{J} = 0. \tag{13b}$$

The momentum equation (12) expresses Newton's second law for stationary ($\bar{u}_0 = 0$) media, and (13) express Maxwell's equations in differential form for stationary media in which regions of electric and magnetic polarization may exist.

The static quantities, which will be distinguished by the superscript or subscript "0," are real functions of the space coordinates. All other quantities, except the real frequency ω, represent complex, space-dependent, harmonic amplitudes, which superimpose upon the static variables as products with $e^{-i\omega t}$. For example, if the total stress dyadic at a point in space and time is denoted by $\bar{\bar{T}}'(\bar{r}, t)$ then $\bar{\bar{T}}'$ can be divided into a real static part $\bar{\bar{T}}_0(\bar{r})$ and a complex harmonic part $\bar{\bar{T}}(\bar{r})e^{-i\omega t}$ i.e.,

$$\bar{\bar{T}}'(\bar{r}, t) = \bar{\bar{T}}_0(\bar{r}) + \bar{\bar{T}}(\bar{r})e^{-i\omega t}.$$

Of course, it is the real part of $\bar{\bar{T}}'$ that corresponds to a measurable quantity. Only the complex harmonic amplitude $\bar{\bar{T}}(\bar{r})$ shows in (12). Similarly the total mass density $\rho_m'(\bar{r}, t)$ may be divided as

$$\rho_m'(\bar{r}, t) = \rho_m^0(\bar{r}) + \rho_m(\bar{r})e^{-i\omega t},$$

but only the real static part occurs in (12). In fact, the mass continuity equation is not included in addition to the momentum equation (12) because it merely introduces the extra variable $\rho_m(\bar{r})$.

The other variables in (12) are $\bar{u}$, the harmonic velocity of the material, and $\bar{f}$, the total harmonic volume force. The usual harmonic em field vectors are denoted by $\bar{E}$, $\bar{H}$, $\bar{B}$, $\bar{D}$, and $\bar{J}$. All variables refer to macroscopic quantities which are evaluated, at least in concept, by averaging microscopic variables over small but finite volumes and time intervals.

The number of unknowns in (12) and (13) reduces to the number of equations if the following linear constitutive relations characterize the transducer material:

$$\bar{D} = \bar{\bar{\epsilon}} \cdot \bar{E} + \bar{\bar{\tau}} \cdot \bar{H} + \bar{\bar{\bar{\alpha}}} : \bar{\bar{s}} + \bar{P}_s \tag{14}$$

$$\bar{B} = \bar{\bar{\nu}} \cdot \bar{E} + \bar{\bar{\mu}} \cdot \bar{H} + \bar{\bar{\bar{\beta}}} : \bar{\bar{s}} + \mu_0 \bar{M}_s \tag{15}$$

$$\bar{J} = \bar{\bar{\sigma}} \cdot (\bar{E} + \bar{u} \times \bar{B}_0) + \rho_v^0 \bar{u} + \bar{J}_s \tag{16}$$

$$\bar{\bar{T}} = \bar{\bar{\bar{c}}} : \bar{\bar{s}} - \bar{\bar{\alpha}}' \cdot \bar{E} - \bar{\bar{\beta}}' \cdot \bar{H} \tag{17}$$

$$\bar{f} = \rho_e^0 \bar{E} + \bar{J} \times \bar{B}_0 + \bar{f}_s, \tag{18}$$

where the strain dyadic $\bar{\bar{s}}$ is related to the velocity by

$$\bar{\bar{s}} = -\frac{1}{i\omega}\left(\nabla \bar{u} - \frac{1}{2}(\nabla \times \bar{u}) \times \bar{\bar{I}}\right). \tag{19}$$

($\bar{\bar{I}}$ is the identity dyadic).

The various "dot products" involved in the vector and dyadic transformations of (14–19) can be understood in terms of Cartesian tensor notation, in which $\bar{\bar{\epsilon}} \cdot \bar{E}$, $\bar{\bar{\alpha}}' \cdot \bar{E}$, $\bar{\bar{\alpha}} : \bar{\bar{s}}$ and $\bar{\bar{\bar{c}}} : \bar{\bar{s}}$ become $\epsilon_{ij}E_j$, $\alpha'_{ijk}E_k$, $\alpha_{ijk}s_{jk}$, and $c_{ijkl}s_{kl}$ respectively (summation over repeated dummy indices is implied). The "n-adic" notation [11] is used wherever convenient because it preserves the familiar vector-dyadic notation for the electroacoustic fields [7, 12].

The dyadics $\bar{\bar{\sigma}}$, $\bar{\bar{\epsilon}}$, $\bar{\bar{\mu}}$, and $(\bar{\bar{\tau}}, \bar{\bar{\nu}})$ stand for the conductivity, permittivity, permeability, and magneto-electric properties [13] of the transducer material.

The direct piezoelectric and piezomagnetic stress triadics [14], $\bar{\bar{\bar{\alpha}}}$ and $\bar{\bar{\bar{\beta}}}$, may be chosen symmetric:

$$\alpha_{ijk} = \alpha_{ikj} \tag{20a}$$

$$\beta_{ijk} = \beta_{ikj}, \tag{20b}$$

since the strain dyadic is symmetric,

$$s_{jk} = \frac{-1}{2i\omega}\left\{\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right\} = s_{kj}. \tag{21}$$

Similarly, because the stress is a symmetric dyadic ($T_{ij} = T_{ji}$) (provided there are no distributions of body torque, stress couples, or angular momentum density [15]), the converse piezoelectric and piezomagnetic stress triadics [14], $\bar{\bar{\bar{\alpha}}}'$ and $\bar{\bar{\bar{\beta}}}'$, must possess the symmetry,

$$\alpha'_{ijk} = \alpha'_{jik} \tag{22a}$$

$$\beta'_{ijk} = \beta'_{jik}. \tag{22b}$$

The symmetry of the stress dyadic also requires the Hooke's tetradic $\bar{\bar{\bar{c}}}$ to obey the symmetry

$$c_{ijkl} = c_{jikl}, \tag{23a}$$

and the symmetry of the strain dyadic allows the defined symmetry

$$c_{ijkl} = c_{ijlk}. \tag{23b}$$

The static charge distribution ρ_e^0, the static magnetic induction $\bar{B}_0$, all other static quantities, and material parameters which may depend upon frequency ω, remain independent of the values of the harmonic variables.

Although only transducers which obey the homogeneous equations are considered finally (source-free transducers), the harmonic sources of polarization $\bar{P}_s$, magnetization $\bar{M}_s$ (μ_0 is the permeability of free-space), current $\bar{J}_s$, and volume force $\bar{f}_s$ have been included in (14), (15), (16), and (18) to allow insight into applications involving these sources.

A number of other assumptions underlie the development of (14–19). In the current equation (16) the "Hall effect" of the magnetic induction on drift velocity is ignored, as well as harmonic variations in conductivity which might be found, for example, in a biased carbon microphone.

The volume forces on electric and magnetic dipoles, static currents, and harmonic variations in mass density (acted upon by gravity) have been omitted in the force equation (18). The effects of gravity are usually minute, and if the material that contains the dipoles or static currents is rigid or has insignificant effect on the essential behavior of the transducer, the volume forces on the dipoles and static currents are justifiably neglected also. The magnet in a dynamic speaker may experience appreciable volume forces, but because the magnet is relatively rigid, (12–19) describe such speakers. However, the operation of a hypothetical transducer containing dipole or static current elements which move through strongly inhomogeneous em fields would not, in general, be described by (18).

Application of Maxwell's stress tensor reveals that forces on surface charges and currents can be evaluated by substituting the appropriate surface delta functions into (18) if nonconductors contain no static surface charge, and if the electric field just outside the surface of conductors, on which static surface charge exists, is used in the evaluation of $\rho_e^0\bar{E}$. The force term $\rho_e\bar{E}_0$ is not needed in (18) because it vanishes in nonconductors where the harmonic charge distribution ρ_e must be zero, and in conductors where the static electric field $\bar{E}_0$ must be zero. The standard condenser microphone is well described by (12–19).

The constitutive equations (14–19) may be used to reduce the three physical laws (12), (13a), and (13b) to functions of three vectors, the electric field $\bar{E}$, the magnetic field $\bar{H}$, and the velocity field $\bar{u}$. The final three equations are conveniently displayed in operator matrix form as

$$\begin{bmatrix} (i\omega\bar{\bar{\epsilon}}\cdot - \bar{\bar{\sigma}}\cdot) & (\nabla\times + i\omega\bar{\bar{\tau}}\cdot) & (-\bar{\bar{\alpha}}:\nabla + (\bar{\bar{\sigma}}\times\bar{B}_0)\cdot - \rho_e^0) \\ (\nabla\cdot\times - i\omega\bar{\bar{\nu}}\cdot) & (-i\omega\bar{\bar{\mu}}\cdot) & (\bar{\bar{\beta}}:\nabla) \\ (\nabla\cdot\bar{\bar{\alpha}}'\cdot + \bar{B}_0\times\bar{\bar{\sigma}}\cdot - \rho_e^0) & (\nabla\cdot\bar{\bar{\beta}}'\cdot) & \left(\dfrac{\nabla\cdot\bar{\bar{c}}:\nabla}{i\omega} - (\bar{B}_0\times\bar{\bar{\sigma}}\times\bar{B}_0)\cdot + \rho_e^0\bar{B}_0\times - i\omega\rho_m^0\right) \end{bmatrix} \begin{bmatrix} \bar{E} \\ \bar{H} \\ \bar{u} \end{bmatrix} =$$

$$\begin{bmatrix} \bar{J}_s - i\omega\bar{P}_s \\ i\omega\mu_0\bar{M}_s \\ \bar{f}_s - \bar{B}_0\times\bar{J}_s \end{bmatrix} \tag{24}$$

Since $(\nabla\times\bar{u})\times\bar{\bar{I}}$ is antisymmetric, its dot products with the symmetric $\bar{\bar{\alpha}}$, $\bar{\bar{\beta}}$, and $\bar{\bar{c}}$ vanish. All operator symbols of the 3×3 matrix containing three vectors are evaluated in succession from right to left. In full scalar matrix notation, the left side of (24) becomes a 9×9 matrix operating on a 9-element column matrix, and the right side another 9-element column matrix.

The particular arrangement of elements in the matrix of (24) was chosen because it exhibits Maxwell's equations as a 2×2 submatrix (the upper left corner), and because the associated adjoint operator matrix also represents the equations of an electroacoustic transducer (the adjoint transducer).

4. The Adjoint Operator Matrix and an Electroacoustic Reciprocity Lemma

The development of the desired generalized reciprocity relations requires the intermediate derivation of a bilinear divergence expression which combines, essentially, the Lorentz lemma [16]

of electromagnetic theory with the acoustical reciprocity theorem ((45) below). The theory of adjoint operators assures the existence of such a bilinear expression (called the "bilinear identity"), and provides a straightforward method for obtaining the identity from the operator matrix of (24) [17]. Specifically, if L denotes the linear operator matrix of (24), the bilinear identity may be written in differential form as

$$\phi_{1t} L \phi_2 - \phi_{2t} L^a \phi_1 = \nabla \cdot \bar{F}, \tag{25}$$

where L^a denotes the adjoint operator associated with L; ϕ_1 and ϕ_2 are any two continuously differentiable column matrices containing three vectors (the subscript t transposes the column matrix to the corresponding row matrix); and $\nabla \cdot \bar{F}$ is the required divergence expression. The explicit evaluation of the bilinear concomitant $\bar{F}$ requires the determination of the adjoint operator L^a, which is actually defined by (25) [17].

For a given linear operator L, the adjoint operator L^a exists uniquely [9], and may be found by transposing the matrix elements of L, and replacing the differential operators by their adjoints.[4] Fortunately, the adjoints of the four different linear differential operators of L,

$$\nabla \times, \bar{\bar{\alpha}} : \nabla, \nabla \cdot \bar{\bar{\alpha}}' \cdot, \text{and} \nabla \cdot \bar{\bar{\bar{c}}} : \nabla, \tag{26a, b, c, d}$$

are readily determined as

$$\nabla \times, -\nabla \cdot \bar{\bar{\alpha}}_t \cdot, -\bar{\bar{\alpha}}_t' : \nabla, \text{and} \nabla \cdot \bar{\bar{\bar{c}}}_t : \nabla, \tag{27a, b, c, d}$$

respectively, with the help of the associated identities,

$$\bar{A}_1 \cdot (\nabla \times \bar{A}_2) - \bar{A}_2 \cdot (\nabla \times \bar{A}_1) = \nabla \cdot (\bar{A}_2 \times \bar{A}_1) \tag{28a}$$

$$\bar{A}_1 \cdot (\nabla \cdot (\bar{\bar{\alpha}} \cdot \bar{A}_2)) + \bar{A}_2 \cdot (\bar{\bar{\alpha}}_t : \nabla \bar{A}_1) = \nabla \cdot (\bar{A}_1 \cdot \bar{\bar{\alpha}} \cdot \bar{A}_2) \tag{28b}$$

$$\bar{A}_1 \cdot (\nabla \cdot (\bar{\bar{\bar{c}}} : \nabla \bar{A}_2)) - \bar{A}_2 \cdot (\nabla \cdot (\bar{\bar{\bar{c}}}_t \pm \nabla \bar{A}_1)) = \nabla \cdot (\bar{A}_1 \cdot \bar{\bar{\bar{c}}} : \nabla \bar{A}_2 - \bar{A}_2 \cdot \bar{\bar{\bar{c}}}_t : \nabla \bar{A}_1). \tag{28c}$$

The subscript t denotes the transposed matrix, and $\bar{A}_1, \bar{A}_2$ are arbitrary, continuously differentiable vectors. In tensor notation the transpose of α_{ijk} and c_{ijkl}, for example, used in (27), (28) and below is α_{jki} and c_{klij}. Note that the curl operator is self-adjoint, that the direct and converse piezoelectric and piezomagnetic operators are adjoints of each other if $\bar{\bar{\alpha}}_t = -\bar{\bar{\alpha}}'$, $\bar{\bar{\beta}} = -\bar{\bar{\beta}}_t'$, and that the purely acoustic operator is self-adjoint if $\bar{\bar{\bar{c}}}_t = \bar{\bar{\bar{c}}}$. These self-adjoint properties are understandable from Lanczos' theory of adjoint operators where he shows that the curl operator is self-adjoint and that the gradient and divergence operators are negative adjoints of each other [17].

The straightforward transposition of the nondifferential elements of L completes the construction of the adjoint operator L^a:

$$L^a = \begin{bmatrix} (i\omega \bar{\bar{\epsilon}}_t \cdot - \bar{\bar{\sigma}}_t \cdot) & (\nabla \times - i\omega \bar{\bar{\nu}}_t \cdot) & -\bar{\bar{\alpha}}_t' : \nabla - \bar{\bar{\sigma}}_t \times \bar{B}_0) \cdot - \rho_e^0 \\ (\nabla \cdot \bar{\bar{\alpha}}_t \cdot - \bar{B}_0 \times \bar{\bar{\sigma}}_t \cdot - \rho_e^0) & (-\nabla \cdot \bar{\bar{\beta}}_t \cdot) & \left[\frac{\nabla \cdot \bar{\bar{\bar{c}}}_t : \nabla}{i\omega} - (\bar{B}_0 \times \bar{\bar{\sigma}}_t \times \bar{B}_0) \cdot - \underset{e}{\rho^0} \bar{B}_0 \times - i\omega \rho_m^0 \right] \end{bmatrix} . \tag{29}$$

Comparison of the adjoint operator L^a with L reveals that, at a given frequency ω, the adjoint

[4] Determination of adjoint differential operators as implied by the bilinear identity (25) is not synonymous with the interchanging of rows and columns of the Cartesian matrix form of the operator. Contrast, for example, Lanczos, ref. [17] with Auld, ref. [12].

operator describes the equations of a second transducer (the adjoint transducer) in which the transposed parameters

$$\bar{\bar{\sigma}}_t,\ \bar{\bar{\epsilon}}_t,\ \bar{\bar{\mu}}_t,\ -\bar{\bar{\tau}}_t,\ -\bar{\bar{\nu}}_t,\ \bar{\bar{\alpha}}_t,\ -\beta_t,\ \bar{\bar{\alpha}}'_t,\ -\bar{\bar{\beta}}'_t,\ \bar{\bar{c}}_t,\ -\bar{B}_0,\ \rho_v^0,\ \text{and}\ \rho_m^0 \tag{30a}$$

replace the original parameters

$$\bar{\bar{\sigma}},\ \bar{\bar{\epsilon}},\ \bar{\bar{\mu}},\ \bar{\bar{\nu}},\ \bar{\bar{\tau}},\ \bar{\bar{\alpha}}',\ \bar{\bar{\beta}}',\ \bar{\bar{\alpha}},\ \bar{\bar{\beta}},\ \bar{\bar{c}},\ \bar{B}_0,\ \rho_v^0\ \text{and}\ \rho_m^0, \tag{30b}$$

respectively. The adjoint transducer refers to a mathematically conceived transducer with hypothetical material parameters which may or may not be physically realizable (see sec. 7). When the transposed parameters (30a) equal the original parameters (30b), the operators L and L^a become identical to form a self-adjoint operator for which the adjoint transducer and original transducer are one and the same. In that case $\bar{B}_0$ is zero ($\bar{B}_0 = -\bar{B}_0$), which corresponds to the electric-type coupling of Primakoff and Foldy [7].

The expression (29) for the adjoint operator allows the explicit evaluation of $\nabla \cdot \bar{F}$. If ϕ_2 is restricted to solutions of (24), and ϕ_1 to solutions of the adjoint equations corresponding to (24) with L^a replacing L, i.e.,

$$L\phi_2 = Q,\ L^a\phi_1 = Q^a \tag{31a,b}$$

where

$$\phi_2 = \begin{bmatrix} \bar{E} \\ \bar{H} \\ \bar{u} \end{bmatrix},\quad \phi_1 = \begin{bmatrix} \bar{E}^a \\ \bar{H}^a \\ \bar{u}^a \end{bmatrix} \tag{32a,b}$$

and

$$Q = \begin{bmatrix} \bar{J}_s - i\omega\bar{P}_s \\ i\omega\mu_0\bar{M}_s \\ \bar{f}_s - \bar{B}_0 \times \bar{J}_s \end{bmatrix},\ Q^a = \begin{bmatrix} \bar{J}_s^a - i\omega\bar{P}_s^a \\ i\omega\mu_0\bar{M}_s^a \\ \bar{f}_s^a + \bar{B}_0 \times \bar{J}_s^a \end{bmatrix} \tag{33a,b}$$

(33a,b)

then $\nabla \cdot \bar{F}$ becomes, after rearrangement and use of (17) and (19),

$$\begin{aligned} \nabla \cdot \bar{F} &= \nabla \cdot (\bar{E} \times \bar{H}^a - \bar{E}^a \times \bar{H}) - \nabla \cdot (\bar{u}^a \cdot \bar{\bar{T}} - \bar{u} \cdot \bar{\bar{T}}^a) \\ &= \bar{E}^a \cdot (\bar{J}_s - i\omega\bar{P}_s) + i\omega\mu_0\bar{H}^a \cdot \bar{M}_s + \bar{u}^a \cdot (\bar{f}_s - \bar{B}_0 \times \bar{J}_s) \\ &\quad - \bar{E} \cdot (\bar{J}_s^a - i\omega\bar{P}_s^a) - i\omega\mu_0\bar{H} \cdot \bar{M}_s^a - \bar{u} \cdot (\bar{f}_s^a + \bar{B}_0 \times \bar{J}_s^a). \end{aligned} \tag{34}$$

When the harmonic sources are set equal to zero (in the transducer not the power supply), application of the divergence theorem to (34) in the volume bounded by the surface A_0 of the transducer yields a generalized reciprocity "lemma" between the independent electroacoustic fields of a given transducer and those of its mathematical adjoint:

$$\int_{A_0} (\bar{E} \times \bar{H}^a - \bar{E}^a \times \bar{H}) \cdot \hat{n} da + \int_{A_0} (\bar{\bar{T}}^a \cdot \bar{u} - \bar{\bar{T}} \cdot \bar{u}^a) \cdot \hat{n} da = 0. \tag{35}$$

The name "generalized electroacoustic reciprocity lemma" seems appropriate for (35) since it represents a merger of the Lorentz lemma [16] of em theory and the acoustical reciprocity theorem ((45) below) as applied to adjoint systems. For self-adjoint transducers, i.e , transducers for which the transposed material parameters of (30a) equal those of (30b), (35) reduces to the

corresponding Primakoff-Foldy expression [7] in which a second set of fields on the same transducer replace the adjoint fields. Auld [12] uses this latter expression with the complex conjugate fields to derive the orthogonality relation [18] between different modes of piezoelectric waveguides and resonators. Equation (35) demonstrates that the waveguides and resonators need not be restricted to piezoelectric material in order for the modes to satisfy orthogonality.

A more general expression similar to (35) may be derived from the equations represented by the operator L. Each row of the operator L in (24) could be multiplied by an arbitrary constant without affecting the essential content of the equations. Specifically, if the first, second, and third row of L are multiplied by arbitrary complex constants, c_1, c_2, and c_3 respectively, (35) is replaced by

$$\int_{A_0} [c_1(\bar{E}\times\bar{H}^a)-c_2(\bar{E}^a\times\bar{H})]\cdot\hat{n}da+c_3\int_{A_0}(\bar{\bar{T}}^a\cdot\bar{u}-\bar{\bar{T}}\cdot\bar{u}^a)\cdot\hat{n}da=0. \tag{36}$$

Although (36) has mathematical validity and interest, the constants c_1, c_2, and c_3 must be restricted if (36) is to be used in the derivation of the generalized reciprocity relations between a transducer and its adjoint transducer. (The adjoint fields will depend upon c_1, c_2, c_3.) First, because the right side of (36) is zero, the constant c_1, can be set equal to unity without loss of generality. Second, c_2 must equal $c_1=1$ to accomplish the derivation of the scattering-matrix reciprocity relations ((53) below). And third, if the adjoint operator underlying (36) is required to describe the fields of a second transducer (the adjoint transducer), comparison of (29) with (24) (when the constants c_1, c_2, and c_3 are included) shows that for $c_1=c_2=1$, the only allowable values of c_3 are ± 1 unless both $\bar{B}_0$ and ρ_e^0 are zero. If both $\bar{B}_0$ and ρ_e^0 vanish throughout the transducers, the adjoint operator represents an adjoint transducer for all values of c_3.

Equation (35) already represents the case of $c_3=+1$, where the relationship between the adjoint transducer parameters and those of the original transducer is given by (30).

For $c_3=-1$, the operator L in (24) changes to L^-, the superscript $(-)$ indicating that the third row of L has been multiplied by a minus sign. The operator adjoint to L^- describes an adjoint transducer in which the transposed parameters

$$\bar{\bar{\sigma}}_t,\ \bar{\bar{\epsilon}}_t,\ \bar{\bar{\mu}}_t,\ -\bar{\bar{\tau}}_t,\ -\bar{\bar{\nu}}_t,\ -\bar{\bar{\alpha}}_t,\ \bar{\bar{\beta}}_t,\ -\bar{\bar{\alpha}}_t',\ \bar{\bar{\beta}}_t',\ \bar{\bar{c}}_t,\ \bar{B}_0,\ -\rho_e^0,\ \text{and}\ \rho_m^0, \tag{37}$$

respectively, replace the original parameters listed in (30b). The transducer of (30a) ($c_3=1$) converts to the transducer of (37) ($c_3=-1$), and vice versa, by reversing the signs of the off-diagonal elements in the third row and column of L^a.

When the transposed parameters (37) assume the values of the original parameters (30b), the operator L^- and its adjoint L^{-a} become identical to form a self-adjoint operator for which the adjoint and original transducer are one and the same. In that case, ρ_e^0 must be zero ($\rho_e^0=-\rho_e^0$), which corresponds to the magnetic-type coupling of Primakoff and Foldy [7].

The equations represented by the operators L and L^- describe the same transducer. However, the equations represented by the adjoints L^a and L^{-a} refer, in general, to different adjoint transducers. Section 5 shows that a transducer self-adjoint with respect to L satisfies reciprocity, while a transducer self-adjoint with respect to L^- satisfies antireciprocity.

5. Generalized Reciprocity Relations

The derivation of the desired reciprocity relations can now be accomplished with the help of the reciprocity lemma (36). It will also be demonstrated that the *SIM* relations (9) and *PWSM* eqs (8) result as a by-product of the reciprocity derivation. Such a demonstration is included simply to prove that the two matrix descriptions of a transducer, which satisfies eqs (12–19), need not be postulated as a definition of linearity, but follow from the linearity of (12–19).

The reciprocity lemma (36) for adjoint transducers may be written as

$$\int_{A_0} [(\bar{E}^a \times \bar{H}) - (\bar{E} \times \bar{H}^a)] \cdot \hat{n} da \pm \int_{A_0} (p^a u_n - p u_n^a)\, da = 0, \tag{38}$$

where the plus sign refers to the operator L and the minus sign to L^-. The surface A_0 of the transducer has been taken in the surrounding fluid, so that the stress tensor relates to pressure like

$$-\hat{n} \cdot \bar{\bar{T}} \cdot \hat{n} = p$$

$$-\hat{n} \cdot \bar{\bar{T}}^a \cdot \hat{n} = p^a,$$

except on the feed area S_0 where the contribution to the integral is assumed negligible.

The integral over A_0 of the em fields expands to

$$V_0^a I_0 - V_0 I_0^a + \int_{A_0 - S_0} [(\bar{E}^a \times \bar{H}) - (\bar{E} \times \bar{H}^a)] \cdot \hat{n} da \tag{39}$$

($\hat{n}$ points into the transducer), after substitutions of the modal fields normalized to unity by (7). The contribution from the integral over $A_0 - S_0$, the area of the transducer excluding the area S_0, vanishes if either of the following conditions is satisfied:

(1) The surface $A_0 - S_0$ of the transducer (but not necessarily the surface $A_p - S_0$ of the power supply) is electromagnetically shielded so that $\bar{E}$, $\bar{E}^a$ or $\bar{H}$, $\bar{H}^a$ are zero on $A_0 - S_0$. (Prescribed boundary conditions on the adjoint transducer are chosen to follow those of the original transducer.)

(2) The surface $A_p - S_0$ of the power supply (but not necessarily the surface $A_0 - S_0$ of the transducer) is electromagnetically shielded and no em sources exist external to the transducer-plus-(power supply or detector) A. (Regions of linear em material are permitted outside A.) Under these conditions, the surface $A_0 - S_0$ may be enlarged to a surface of infinite radius where the Sommerfeld radiation condition or the exponential decay in electromagnetically lossy media demands that the integral be zero.

If the transducer operates at quasi-static frequencies so that the em fields outside the transducer obey

$$\bar{E} = -\nabla \psi, \bar{E}^a = -\nabla \psi^a \tag{40a}$$

$$\nabla \times \bar{H} = \bar{J}, \nabla \times \bar{H}^a = \bar{J}^a, \tag{40b}$$

modal theory may not apply, but substitution of (40) into the em integral of (38) shows that the contribution over $A_0 - S_0$ still vanishes. That is, no shielding is required at quasi-static frequencies of operation for the expression

$$V_0^a I_0 - V_0 I_0^a$$

to represent the em term of (38).

With the em integral over $A_0 - S_0$ zero, (38) becomes

$$V_0^a I_0 - V_0 I_0^a \pm \int_{A_0} (p^a u_n - p u_n^a)\, da = 0. \tag{41}$$

Because the fields of the transducer and its adjoint exist independently and the velocity-current variables of each transducer may assume arbitrary values, (41) transforms into the *SIM* relations by first choosing

$$I_{01}^a = 1, \qquad u_{n1}^a(\bar{r}) = 0, \tag{42a}$$

which gives

$$V_0 = V_{01}^a I_0 \pm \int_{A_0} p_1^a(\bar{r}) u_n(\bar{r})\, da; \tag{43a}$$

then choosing

$$I_{02}^a = 0, u_{n2}^a(\bar{r}) = \delta(\bar{r} - \bar{r}'), \tag{42b}$$

which gives

$$p(\bar{r}) = \pm V_{02}^{a}(\bar{r}) I_0 + \int_{A_0} p_2^{a}(\bar{r},\bar{r}_0) u_n(\bar{r}_0)\, da_0. \tag{43b}$$

Comparison of (43) with (9) shows that the various spatial impedances equate as

$$\begin{aligned} Z_b &= V_{01}^{a}, & h'(\bar{r}) &= \pm p_1^{a}(\bar{r}), \\ h(\bar{r}) &= \pm V_{02}^{a}(\bar{r}), & Z_0(\bar{r},\bar{r}_0) &= p_2^{a}(\bar{r},\bar{r}_0). \end{aligned} \tag{44}$$

The *PWSM* equations (8) are disclosed in a similar manner once (2, 3, 6) and the acoustical reciprocity theorem,

$$\int_A (p^1 u_n^2 - p^2 u_n^1)\, da + \int_{S_1} (p^1 u_z^2 - p^2 u_z^1)\, d\bar{R} = 0, \tag{45}$$

as applied to any two acoustic fields supported by the fluid between A and S_1, convert (41) to

$$\eta_0 (b_0 a_0^{a} - b_0^{a} a_0) \pm \int_{\bar{K}} \eta(K) [a(\bar{K}) b^{a}(-\bar{K}) - a^{a}(-\bar{K}) b(\bar{K})]\, d\bar{K} = 0. \tag{46}$$

Since the values of a_0, a and a_0^{a}, a^{a} may be designated arbitrarily and independently of each other, choose

$$a_{01}^{a} = 1,\ a_1^{a}(\bar{K}) = 0, \tag{47a}$$

and

$$a_{02}^{a} = 0,\ a_2^{a}(\bar{K}) = \delta(\bar{K} + \bar{L}), \tag{47b}$$

to extract the *PWSM* equations from (46),

$$b_0 = b_{01}^{a}\, a_0 \mp \frac{1}{\eta_0} \int_{\bar{L}} \eta(L)\, b_1^{a}(-\bar{L}) a(\bar{L})\, d\bar{L} \tag{48a}$$

$$b(\bar{K}) = \mp \frac{\eta_0}{\eta(K)} b_{02}^{a}(\bar{K}) a_0 + \frac{1}{\eta(K)} \int_L \eta(L) b_2^{a}(\bar{K},-\bar{L}) a(\bar{L})\, d\bar{L}. \tag{48b}$$

Equations (48) coincide with (8) under the replacements,

$$S_{00} = b_{01}^{a},\ S_{01}(\bar{L}) = \mp \eta(L) b_1^{a}(-\bar{L})/\eta_0, \tag{49}$$

$$S_{10}(\bar{K}) = \mp \eta_0 b_{02}^{a}(\bar{K})/\eta(K),\ S_{11} = \eta(L) b_2^{a}(\bar{K},-\bar{L})/\eta(K).$$

If the conditions of (42) and (47) are applied to the given transducer rather than its adjoint transducers, (41) and (46) produce (43) and (48) with the given transducer and adjoint transducers interchanged. That is, the *SIM* relations and the *PWSM* equations describe the adjoint transducers as well as the original transducer.

The generalized reciprocity relations between a given transducer and its adjoint transducers may also be extracted from (41) and (46) by eliminating the pressures and voltages of (41) with the *SIM* relations, and the outgoing amplitudes of (46) with the *PWSM* equations. Such a procedure yields

$$I_0^a(Z_b^a - Z_b)I_0 + I_0 \int_{A_0} [h^{a\prime}(\bar{r}) \mp h(\bar{r})] u_n^a(\bar{r})\, da - I_0^a \int_{A_0} [h'(\bar{r}) \mp h^a(\bar{r})] u_n(\bar{r})\, da$$

$$\pm \int_{A_0} \int_{A_0} u_n^a(\bar{r}) [Z_0^a(\bar{r}_0, \bar{r}) - Z_0(\bar{r}, \bar{r}_0)] u_n(\bar{r}_0) da_0\, da = 0 \qquad (50)$$

for (41), and

$$\eta_0 a_0^a (S_{00}^a - S_{00}) a_0 + a_0 \int_{\bar{L}} [\eta_0 S_{01}^a(\bar{L}) \mp \eta(L) S_{10}(-\bar{L})] a^a(\bar{L})\, d\bar{L}$$

$$+ a_0^a \int_{\bar{L}} [\eta_0 S_{01}(\bar{L}) \mp \eta(L) S_{10}^a(-\bar{L})] a(\bar{L})\, d\bar{L} \qquad (51)$$

$$\pm \int_{\bar{L}} \int_{\bar{K}} a^a(\bar{L}) [\eta(K) S_{11}^a(-\bar{K}, \bar{L}) - \eta(L) S_{11}(-\bar{L}, \bar{K})] a(\bar{K})\, d\bar{K} d\bar{L} = 0$$

for (46).

Since they must hold for arbitrary values of the inputs, (50) and (51) imply

$$Z_b^a = Z_b \qquad (52a)$$

$$h^{a\prime}(\bar{r}) = \pm h(\bar{r}) \qquad (52b)$$

$$h^a(\bar{r}) = \pm h'(\bar{r}) \qquad (52c)$$

$$Z_0^a(\bar{r}_0, \bar{r}) = Z_0(\bar{r}, \bar{r}_0) \qquad (52d)$$

and

$$S_{00}^a = S_{00} \qquad (53a)$$

$$\eta_0 S_{01}^a(\bar{K}) = \pm \eta(K) S_{10}(-\bar{K}) \qquad (53b)$$

$$\eta(K) S_{10}^a(\bar{K}) = \pm \eta_0 S_{01}(-\bar{K}) \qquad (53c)$$

$$\eta(K) S_{11}^a(-\bar{K}, \bar{L}) = \eta(L) S_{11}(-\bar{L}, \bar{K}). \qquad (53d)$$

Either the "*b*" or "*c*" equation of (52) and (53) may be considered redundant since the entire derivation could be repeated with the given and adjoint transducers interchanged.

It is (53b) that enables the characteristic receiving matrix S_{01} of an arbitrary linear transducer to be expressed mathematically as a function of the equivalent sources within the adjoint transducers [19]. As mentioned in the introduction, such expressions have special utility in the extrapolation techniques of the *PWSM* formulation [4, 8].

When a transducer is self-adjoint with respect to L or L^-, (52) and (53) reduce to the original reciprocity relations of (10) and (11). Reciprocity defined by both Kerns and Foldy-Primakoff is satisfied by an electroacoustic transducer that exhibits electric-type coupling (e.g., the condenser microphone) to form a self-adjoint operator L. Antireciprocity is satisfied by a transducer that exhibits magnetic-type coupling (e.g., the dynamic speaker) to form a self-adjoint operator L^-. If the static fields as well as the piezoelectric and piezomagnetic coupling coefficients vanish, comparison of (30a) with (37) reveals that both adjoint transducers become identical to the original transducer, and by (52) and (53), the transmitting and receiving parameters h, h', S_{10}, S_{01} also vanish. In other words, there would be no electroacoustic coupling—a result that the constitutive relations (14–18) confirm.

6. Adjoint Reciprocity Theorems

The generalized reciprocity relations (52) and (53) may find expression in a variety of derivative "reciprocity theorems and principles." As a consequence of (52) or (53), the "electroacoustic reciprocity theorem" [6, 20] for reciprocal transducers extends to nonreciprocal transducers by the relating of the microphone response of the given transducer to the speaker response of its adjoint transducers (and vice versa). Similarly, the "principle of reciprocity" [21] that relates scattered pressure at a point B from a point source at A to scattered pressure at point A when the point source is moved to B may be extended to nonreciprocal transducers through the concept of the adjoint transducer. The following derivation of the aforementioned results are performed easily and in great generality by using the Kerns *PWSM* description.

The standard speaker response $S(\bar{r})$ of a given transducer is defined as the ratio of the pressure at a point r in the ambient fluid to the transducer input current I_0 [6],

$$S(\bar{r}) \equiv \frac{p(\bar{r})}{I_0}. \tag{54}$$

Because incoming acoustic waves are assumed zero, i.e., $a(K)=0$, the pressure in (2a) may be expressed in terms of $b(K)=V(K)$ alone,

$$p(\bar{r}) = \frac{1}{2\pi}\int_{\bar{K}} b(\bar{K}) e^{i\gamma z} e^{i\bar{K}\cdot\bar{r}}\, d\bar{K}. \tag{55}$$

The factor $e^{i\gamma z}$ appearing in the integral of (55) allows the z-coordinate of $\bar{r}$ to assume values other than zero. (The fluid must contain no sources or inhomogeneities between $z=0$ and the z in (55).) The *PWSM* equations (8), which reduce to

$$b_0 = S_{00} a_0 \tag{56a}$$

$$b(\bar{K}) = S_{10}(\bar{K}) a_0 \tag{56b}$$

when $a(\bar{K})=0$, combine with (6b) to yield the following expression for $b(\bar{K})$ in terms of I_0:

$$b(\bar{K}) = \frac{S_{10}(\bar{K}) I_0}{(1-S_{00})\eta_0}. \tag{57}$$

After substitution of $b(\bar{K})$ from (57) into (55), the speaker response (54) takes the final form of

$$S(\bar{r}) = \frac{1}{2\pi(1-S_{00})\eta_0}\int_{\bar{k}} S_{10}(\bar{K}) e^{i\bar{k}\cdot\bar{r}}\, d\bar{K},\ \bar{k}=\bar{K}+\gamma\hat{e}_z. \tag{58}$$

The adjoint microphone response is defined as the ratio of the open-circuit voltage V_0 for an adjoint transducer in the presence of a spherical pressure wave centered at point $\bar{r}$ in the fluid to the pressure in the spherical wave at a reference point $\bar{r}_c$ [6],

$$M^a(\bar{r}) \equiv \frac{V_0}{p_r(\bar{r}_c)}, \tag{59}$$

$$p_r(\bar{r}_c) = \frac{\mathscr{A} e^{ik|\bar{r}_c-\bar{r}|}}{|\bar{r}_c-\bar{r}|}. \tag{60}$$

(The amplitude of the spherical wave is denoted by $\mathscr{A}$.) Since I_0 is zero, a_0 equals b_0, and V_0 is equal to $2a_0$. The scattering-matrix equation (8a) for an adjoint transducer relates a_0 to $a(\bar{K})$,

$$a_0 = \frac{1}{(1 - S^a_{00})} \int_{\bar{K}} S^a_{01}(\bar{K}) a(\bar{K}) d\bar{K}. \tag{61}$$

The incoming amplitudes $a(\bar{K})$ are given by (4a) as

$$a(\bar{K}) = V(\bar{K}) = \frac{1}{2\pi} \int_{S_1} p_r(\bar{R}_c) e^{-i\bar{K}\cdot\bar{R}_c} \, d\bar{R}_c. \tag{62}$$

Replacement of the pressure in (62) by its value given in (60) enables the explicit determination of $a(\bar{K})$,

$$a(\bar{K}) = \frac{\mathscr{A} i}{\gamma} e^{i\gamma z} e^{-i\bar{K}\cdot\bar{R}}, \tag{63}$$

which combines with (61), (60), and (59) to produce the final form of the adjoint microphone response,

$$M^a(\bar{r}) = \frac{2id}{(1 - S^a_{00}) e^{ikd}} \int_{\bar{K}} \frac{S^a_{01}(-\bar{K})}{\gamma} e^{i\bar{k}\cdot\bar{r}} \, d\bar{K}, \; d = |\bar{r}_c - \bar{r}|. \tag{64}$$

Since the adjoint reciprocity relations require that

$$S^a_{00} = S_{00}$$

$$\eta_0 S^a_{01}(-\bar{K}) = \frac{\pm\gamma}{\omega\rho_0} S_{10}(\bar{K}),$$

(58) and (64) show that

$$\left| \frac{M^a(\bar{r})}{S(\bar{r})} \right| = \frac{4\pi d}{\omega\rho_0}, \tag{65}$$

or that the magnitude of the ratio of microphone response at $\bar{r}$ of the adjoint transducers to the speaker response at $\bar{r}$ of the original transducer is a constant independent of the position $\bar{r}$ and of the particular transducer involved. The constant could also be made independent of the distance d to the reference point $\bar{r}_c$ if the microphone response were "normalized" to the amplitude $\mathscr{A}$ of the spherical wave. If the transducer is self-adjoint (reciprocal or antireciprocal) (65) becomes identical to the standard electroacoustic reciprocity theorem [20].

Next, consider the derivation of the adjoint "principle of reciprocity" which applies to the scattering of a spherical wave from a given scatterer (or transducer terminated in a passive load). The scattered pressure $p_{sc}(\bar{r}_B, \bar{r}_A)$ at a point $\bar{r}_B$ in the fluid caused by a spherical wave centered at a point $\bar{r}_A$ in the fluid is given, as in (55), by

$$p_{sc}(\bar{r}_B, \bar{r}_A) = \frac{1}{2\pi} \int_{\bar{K}} b(\bar{K}) e^{i\bar{k}\cdot\bar{r}} \, d\bar{K}, \tag{66}$$

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where $b(\bar{K})$ may be found in terms of $a(\bar{K})$ from the scattering equations (8):

$$b(\bar{K}) = \int_{\bar{L}} \left(\frac{S_{10}(\bar{K}) S_{01}(\bar{L}) \Gamma_T}{(1 - \Gamma_T S_{00})} + S_{11}(\bar{K}, \bar{L}) \right) a(\bar{L})\, d\bar{L}. \tag{67}$$

The reflection coefficient Γ_T is related to the terminating load $Z_T = -V_0/I_0$ by

$$\Gamma_T = \frac{Z_T \eta_0 - 1}{Z_T \eta_0 + 1}. \tag{68}$$

Since the source is a spherical pressure wave centered at $\bar{r}_A$, the amplitudes $a(\bar{L})$ take the same form as in (63), i.e.,

$$a(\bar{L}) = \frac{\mathscr{A} i}{\gamma(L)} e^{i\gamma(L) z_A} e^{-i\bar{L} \cdot \bar{R}_A} \tag{69}$$

Substitution of (69) and (67) into (66) results in the scattered pressure

$$p_{sc}(\bar{r}_b, \bar{r}_A) = \frac{\mathscr{A} i}{2\pi} \int_{\bar{K}} \int_{\bar{L}} \frac{\left(\frac{S_{10}(\bar{K}) S_{01}(-\bar{L}) \Gamma_T}{1 - \Gamma_T S_{00}} + S_{11}(\bar{K}, -\bar{L}) \right)}{\gamma(L)} e^{i\bar{K} \cdot \bar{r}_B} e^{i\bar{L} \cdot \bar{r}_A} d\bar{L}\, d\bar{K}. \tag{70}$$

If the position of the source and observer is interchanged and the given transducer is replaced by an adjoint transducer, the scattered pressure at $\bar{r}_A$ is found in a similar manner to be

$$p^a_{sc}(\bar{r}_A, \bar{r}_B) = \frac{\mathscr{A} i}{2\pi} \int_{\bar{K}} \int_{\bar{L}} \frac{\left(\frac{S^a_{10}(\bar{K}) S^a_{01}(-\bar{L}) \Gamma_T)}{1 - \Gamma_T S^a_{00}} + S^a_{11}(\bar{K}, -\bar{L}) \right)}{\gamma(L)} e^{i\bar{K} \cdot \bar{r}_A} e^{i\bar{L} \cdot \bar{r}_B} d\bar{L}\, d\bar{K}. \tag{71}$$

(The termination for the adjoint transducers is kept at Z_T.)

The adjoint reciprocity relations (53) demonstrate from (70) and (71) that

$$p_{sc}(\bar{r}_B, \bar{r}_A) = p^a_{sc}(\bar{r}_A, \bar{r}_B), \tag{72}$$

which completes the proof of the adjoint principle of reciprocity. (Note that the bracketed quantity in the integrands of (70) and (71) represents an equivalent scattering matrix with reciprocal properties identical to S_{11}.) That is, the scattered pressure measured at a point $\bar{r}_B$ from a point source at $\bar{r}_A$ is equal to the scattered pressure measured at the point $\bar{r}_A$ when the point source is moved to $\bar{r}_B$ and an adjoint transducer replaces the given transducer. If the transducer is self-adjoint (reciprocal or antireciprocal) (72) expresses the usual principle of reciprocity [21] for scatterers.

7. Power Relations

In this section power relations are developed and used to show that the dissipative characteristics (lossiness, losslessness, and "gaininess") of the material of the given transducer are preserved pointwise in the adjoint transducers. Such a result lends support to the feasibility of producing the adjoint transducers in the laboratory.

For harmonic fields, the time-average em and acoustic power flow per unit area is given by $\frac{1}{2}\,\mathrm{Re}\,[\bar{E}\times\bar{H}^*]$ and $-\frac{1}{2}\,\mathrm{Re}\,[\bar{\bar{T}}\cdot\bar{u}^*]$ respectively. (The asterisk denotes the complex conjugate.) The total time-average power P flowing into the electroacoustic transducer through its surface A_0 is, therefore, given by

$$P=\frac{1}{2}\,\mathrm{Re}\oint_{A_0}[\bar{E}\times\bar{H}^*-\bar{\bar{T}}\cdot\bar{u}^*]\cdot\hat{n}da, \tag{73}$$

where $\hat{n}$ denotes the unit normal into the transducer. Since the surface A_0 lies in or on the boundary of the ambient fluid,

$$\hat{n}\cdot\bar{\bar{T}}\cdot\bar{u}^*=-pu_n^*,$$

and the power equation (73) becomes

$$P=\frac{1}{2}\,\mathrm{Re}\oint_{A_0}[(\bar{E}\times\bar{H}^*)\cdot\hat{n}+pu_n^*]\,da. \tag{74}$$

If the transducer is electromagnetically shielded, the em part of the integral (74) evaluates simply as $V_0I_0^*$ by expanding $\bar{E}$ and $\bar{H}^*$ in terms of the waveguide mode, i.e., from (5)

$$\bar{E}_t=V_0\bar{e}_0,\ \bar{H}_t^*=I_0^*\,\bar{h}_0,$$

and making use of the normalization equation (7). (Waveguide theory shows that the basis fields $(\bar{e}_0,\bar{h}_0)$ of a propagating mode in a uniform, isotropic guide may be chosen real.)

The *SIM* relations (9) convert the power expression (74) to an Hermitian form containing continuous $(u_n(\bar{r}))$ and discrete (I_0) variables:

$$\begin{aligned}P=&I_0(Z_b+Z_b^*)+\int_{A_0}[I_0(h'^*(\bar{r})+h(\bar{r}))u_n^*(\bar{r})+I_0^*(h'(\bar{r})+h^*(\bar{r}))u_n(\bar{r})]\,da\\&+\int_{A_0}\int_{A_0}u_n(\bar{r}_0)[Z_0(\bar{r},\bar{r}_0)+Z_0^*(\bar{r}_0,\bar{r})]u_n^*(\bar{r})\,da_0\,da.\end{aligned} \tag{75}$$

(The taking of the real part has been accomplished by adding each term to its complex conjugate.)

A transducer may be classified as lossy, lossless, or "gainy" if, for all values of u_n and I_0 (except u_n, I_0 all zero), the power P is positive, zero, or negative, i.e., if the Hermitian form (75) is positive definite, zero, or negative definite [22].

Mercer's theorem [23], which can be generalized to provide necessary and sufficient conditions on the spacial impedances Z_b, h, h', and Z_0 for the power P to belong to one of the "definite" value classes, reduces to very simple relationships among the spacial impedances when the transducer is lossless. In the lossless case, i.e., when the power expression (75) is identically zero for all values of u_n and I_0, various combinations of I_0 equal to zero and $u_n(\bar{r})$ equal to zero or a sum of delta functions can be chosen to prove that the spatial impedances satisfy the following equalities:

$$Z_b+Z_b^*=0 \qquad \text{(i.e., } Z_b \text{ is imaginary)} \tag{76a}$$

$$h'(\bar{r})+h^*(\bar{r})=0 \tag{76b}$$

$$Z_0(\bar{r},\bar{r}_0)+Z_0^*(\bar{r}_0,\bar{r})=0. \tag{76c}$$

A reciprocal or antireciprocal transducer also satisfies (11), which combine with (76) to show that Z_0 (as well as Z_b) is imaginary, and that both h, h' are imaginary or real depending upon whether the transducer is reciprocal or antireciprocal respectively. It can be demonstrated [9] that no simple relations similar to (76) exist among the scattering matrices of lossless transducers.

Substitution of the adjoint relations (52) for the spacial impedances into (75) yields an expression (similar to (75)) which can be manipulated slightly to prove that the power input to both adjoint transducers (one associated with L, the other with L^-) belong to the same value class, i.e., have the same definite or indefinite form, as the power input to the original transducer. However, the power input to the adjoint transducers does not, necessarily remain the same as the power input to the original transducer for given excitation I_0 and $u_n(\bar{r})$.

Finally, consider the problem of expressing the power relation (73) in terms of the internal fields and material parameters of the electroacoustic transducer. The divergence theorem transforms the surface integral (73) into a volume integral over the volume $\mathscr{V}_0$ (enclosed by S_0) of the transducer to produce,

$$P=\int_{\mathscr{V}_0}[\bar{H}^*\cdot(\nabla\times E)+\bar{H}\cdot(\nabla\times\bar{E}^*)-\bar{E}\cdot(\nabla\times\bar{H}^*)-\bar{E}^*\cdot(\nabla\times\bar{H})$$
$$-(\nabla\cdot\bar{\bar{T}})\cdot\bar{u}^*-(\nabla\cdot\bar{\bar{T}}^*)\cdot\bar{u}-\bar{\bar{T}}:\nabla\bar{u}^*-\bar{\bar{T}}^*:\nabla\bar{u}]\,dV, \tag{77}$$

after the divergence is expanded, and the taking of the real part is accomplished by adding each term to its complex conjugate. The equations (12–19) allows the integrand P_r of the power integral (77) to be written as an Hermitian form,

$$P_r=[\bar{H}^*,\bar{E}^*,\bar{\bar{s}}^*,\bar{u}^*]\begin{pmatrix} i\omega(\bar{\bar{\mu}}-\bar{\bar{\mu}}_t^*) & i\omega(\nu-\bar{\bar{\tau}}_t^*) & i\omega(\bar{\bar{\beta}}-\bar{\bar{\beta}}_t'^*) & 0 \\ i\omega(\bar{\bar{\tau}}-\bar{\bar{\nu}}_t^*) & [i\omega(\bar{\bar{\epsilon}}-\bar{\bar{\epsilon}}_t^*)-(\bar{\bar{\sigma}}+\bar{\bar{\sigma}}_t^*)] & i\omega(\bar{\bar{\alpha}}-\bar{\bar{\alpha}}_t') & \bar{B}_0\times(\bar{\bar{\sigma}}+\bar{\bar{\sigma}}_t^*) \\ i\omega(\bar{\bar{\beta}}'-\bar{\bar{\beta}}^*) & i\omega(\bar{\bar{\alpha}}'-\bar{\bar{\alpha}}_t^*) & -i\omega(\bar{\bar{c}}-\bar{\bar{c}}_t^*) & 0 \\ 0 & -\bar{B}_0\times(\bar{\bar{\sigma}}+\bar{\bar{\sigma}}_t^*) & 0 & \bar{B}_0\times(\bar{\bar{\sigma}}+\bar{\bar{\sigma}}_t^*)\times\bar{B}_0 \end{pmatrix}\begin{pmatrix}\bar{H}\\ \bar{E}\\ \bar{\bar{s}}\\ \bar{u}\end{pmatrix} \tag{78}$$

or simply

$$P_r=\tilde{X}^*\mathscr{H}X. \tag{79}$$

P_r may be interpreted as power input per unit volume of the transducer.

A sufficient condition for the total time-average power input to be positive definite, zero, or negative definite is for the Hermitian integrand (78) to be correspondingly positive definite, zero, or negative definite. Likewise, a standard theorem of linear algebra [22] states that the matrix $\mathscr{H}$ must belong to the same value class as the associated Hermitian form (78), provided $\bar{H}$, $\bar{E}$, $\bar{\bar{s}}$ and $\bar{u}$ may assume arbitrary values at a point. In particular, if P_r is zero, the transducer is lossless, $\mathscr{H}$ must be zero and therefore the material parameters satisfy the following simple relationships:

$$\bar{\bar{\sigma}}=-\bar{\bar{\sigma}}_t^*,\ \bar{\bar{\epsilon}}=\bar{\bar{\epsilon}}_t^*,\ \bar{\bar{\mu}}=\bar{\bar{\mu}}_t^*,\ \bar{\bar{\nu}}=\bar{\bar{\tau}}_t^*,\ \bar{\bar{c}}=\bar{\bar{c}}_t^*,\ \bar{\bar{\alpha}}'=\bar{\bar{\alpha}}_t^*,\ \text{and}\ \bar{\bar{\beta}}'=\bar{\bar{\beta}}_t^*. \tag{80}$$

If, in addition, the transducer is reciprocal or antireciprocal, (30) and (37) show that the material parameters also satisfy

$$\bar{\bar{\sigma}}=\bar{\bar{\sigma}}_t,\ \bar{\bar{\epsilon}}=\bar{\bar{\epsilon}}_t,\ \bar{\bar{\mu}}=\bar{\bar{\mu}}_t,\ \bar{\bar{\nu}}=\bar{\bar{\tau}}_t,\ \bar{\bar{c}}=\bar{\bar{c}}_t,\ \bar{\bar{\alpha}}'=\pm\bar{\bar{\alpha}}_t,\ \text{and}\ \bar{\bar{\beta}}'=\mp\bar{\bar{\beta}}_t^*, \tag{81}$$

where the upper and lower sign in the last two equations of (81) refer to reciprocal and antireciprocal transducers respectively. Equations (80) and (81) together imply that a lossless (in the sense of (80)) reciprocal or antireciprocal transducer possesses real $\bar{\bar{\sigma}}$, $\bar{\bar{\epsilon}}$, $\bar{\bar{\mu}}$, $\bar{\bar{c}}$, and imaginary $\bar{\bar{\nu}}$ and $\bar{\bar{\tau}}$. The

piezoelectric triadics $\bar{\bar{\alpha}}, \bar{\bar{\alpha}}'$ are real for reciprocal and imaginary for antireciprocal lossless transducers. Conversely, the piezomagnetic triadics $\bar{\bar{\beta}}, \bar{\bar{\beta}}'$ are imaginary for reciprocal and real for antireciprocal lossless transducers.

Equations (30) and (37) may also be used to demonstrate from (78) and (77) that the total power input for the two transducers adjoint to the original transducer belong to the same value class or have the same dissipative character as the original transducer—a result which concurs with the previous analysis on the spacial impedance power relation (75).

8. Concluding Remarks

A brief summary of the main properties of the L and L^- adjoint transducers is displayed in tables 1–4. From an experimental standpoint, it is naturally of interest to know if the adjoint transducers are physically realizable, i.e., producible in the laboratory. For example, if a transducer adjoint to a given transducer could be produced, the receiving characteristics (S_{01}) of the given transducer could be determined by measuring the transmitting characteristics (S^a_{10}) of the adjoint transducer. Such an indirect experimental procedure would be desirable with a set-up designed to measure transducer far-field patterns.

TABLE 1.

Given transducer (described by eqs (12–19))		L Adjoint transducer	L^- Adjoint transducer
Internal Characteristics		Internal characteristics	Internal characteristics
Conductivity	$\bar{\bar{\sigma}}$	$\bar{\bar{\sigma}}^a = \bar{\bar{\sigma}}_t$	$\bar{\bar{\sigma}}^a = \bar{\bar{\sigma}}_t$
Permittivity	$\bar{\bar{\epsilon}}$	$\bar{\bar{\epsilon}}^a = \bar{\bar{\epsilon}}_t$	$\bar{\bar{\epsilon}}^a = \bar{\bar{\epsilon}}_t$
Permeability	$\bar{\bar{\mu}}$	$\bar{\bar{\mu}}^a = \bar{\bar{\mu}}_t$	$\bar{\bar{\mu}}^a = \bar{\bar{\mu}}_t$
Hooke's tensor	$\bar{\bar{\bar{\bar{c}}}}$	$\bar{\bar{\bar{\bar{c}}}}^a = \bar{\bar{\bar{\bar{c}}}}_t$	$\bar{\bar{\bar{\bar{c}}}}^a = \bar{\bar{\bar{\bar{c}}}}_t$
Magneto-electric tensors	$\bar{\bar{\nu}}, \bar{\bar{\tau}}$	$\bar{\bar{\nu}}^a = -\bar{\bar{\tau}}_t, \bar{\bar{\tau}}^a = -\bar{\bar{\nu}}_t$	$\bar{\bar{\nu}}^a = -\bar{\bar{\tau}}_t, \bar{\bar{\tau}}^a = -\bar{\bar{\nu}}_t$
Direct peizoelectric and piezomagnetic tensors	$\bar{\bar{\alpha}}, \bar{\bar{\beta}}$	$\bar{\bar{\alpha}}^a = \bar{\bar{\alpha}}'_t, \bar{\bar{\beta}}^a = -\bar{\bar{\beta}}_t$	$\bar{\bar{\alpha}}^a = -\bar{\bar{\alpha}}'_t, \bar{\bar{\beta}}^a = \bar{\bar{\beta}}'_t$
Converse piezoelectric and piezomagnetic tensors	$\bar{\bar{\alpha}}', \bar{\bar{\beta}}'$	$\bar{\bar{\alpha}}^{a\prime} = \bar{\bar{\alpha}}_t, \bar{\bar{\beta}}^{a\prime} = -\bar{\bar{\beta}}_t$	$\bar{\bar{\alpha}}^{a\prime} = -\bar{\bar{\alpha}}_t, \bar{\bar{\beta}}^{a\prime} = \bar{\bar{\beta}}_t$
Mass density	ρ^0_m	$\rho^{a0}_m = \rho^0_m$	$\rho^{a0}_m = \rho^0_m$
Static magnetic induction	$\bar{B}_0$	$\bar{B}^a_0 = -\bar{B}_0$	$\bar{B}^a_0 = \bar{B}_0$
Static charge distribution	ρ^0_e	$\rho^{a0}_e = \rho^0_e$	$\rho^{a0}_e = -\rho^0_e$

TABLE 2

Given transducer (described by eqs (12–19))		L Adjoint transducer	L^- Adjoint transducer
External characteristics		External characteristics	External characteristics
Transmitting (*PWSM*)	$S_{10}(\bar{K})$	$S^a_{10}(\bar{K}) = S_{01}(-\bar{K})\eta_0/\eta$	$S^a_{10}(\bar{K}) = -S_{01}(-\bar{K})\eta_0/\eta$
Receiving (*PWSM*)	$S_{01}(\bar{K})$	$S^a_{01}(\bar{K}) = S_{10}(-\bar{K})\eta/\eta_0$	$S^a_{01}(\bar{K}) = -S_{10}(-\bar{K})\eta/\eta_0$
Acoustic scattering (*PWSM*)	$S_{11}(\bar{K}, \bar{L})$	$S^a_{11}(\bar{K}, \bar{L}) = S_{11}(-\bar{L}, -\bar{K})\eta(\bar{L})/\eta(\bar{K})$	$S^a_{11}(\bar{K}, \bar{L}) = S_{11}(-\bar{L}, -\bar{K})\eta(\bar{L})/\eta(\bar{K})$
EM reflection (*PWSM*)	S_{00}	$S^a_{00} = S_{00}$	$S^a_{00} = S_{00}$

TABLE 2—CONTINUED

Given transducer (described by eqs (12–19))		L Adjoint transducer	L^- Adjoint transducer
External characteristics		External characteristics	External characteristics
Speaker transfer (*SIM*)	$h(\bar{r})$	$h^a(\bar{r}) = h'(\bar{r})$	$h^a(\bar{r}) = -h'(\bar{r})$
Microphone transfer (*SIM*)	$h'(\bar{r})$	$h^{a'}(\bar{r}) = h(\bar{r})$	$h^{a'}(\bar{r}) = -h(\bar{r})$
Open-circuit acoustic (*SIM*)	$Z_0(\bar{r}, \bar{r}_0)$	$Z_0^a(\bar{r}, \bar{r}_0) = Z_0(\bar{r}_0, \bar{r})$	$Z_0^a(\bar{r}, \bar{r}_0) = Z_0(\bar{r}_0, \bar{r})$
Blocked electrical (*SIM*)	Z_b	$Z_b^a = Z_b$	$Z_b^a = Z_b$

TABLE 3

Given transducer (described by eqs (12–19))		L Adjoint transducer	L^- Adjoint transducer
Power input		Power input	Power input
Passive transducers	$p > 0$ $p \geq 0$ $p = 0$	$p^a > 0$ $p^a \geq 0$ $p^a = 0$	$p^a > 0$ $p^a \geq 0$ $p^a = 0$
Active transducers	$p < 0$ $p \leq 0$ Indefinite	$p^a < 0$ $p^a \leq 0$ Indefinite	$p^a < 0$ $p^a \leq 0$ Indefinite

TABLE 4

Given transducer (described by eqs (12–19))		L Adjoint transducer	L^- Adjoint transducer
Adjoint reciprocity theorems		Adjoint reciprocity theorems	Adjoint Reciprocity theorems
Speaker Resposne	$S(\bar{r})$	$S^a(\bar{r}) = M(\bar{r})\omega\rho_0 e^{ikd}/4\pi id$	$S^a(\bar{r}) = -M(\bar{r})\omega\rho_0 e^{ikd}/4\pi id$
Microphone response	$M(\bar{r})$	$M^a(\bar{r}) = S(\bar{r})4\pi id/\omega\rho_0 e^{ikd}$	$M^a(\bar{r}) = -S(\bar{r})4\pi id/\omega\rho_0 e^{ikd}$
Scattered pressure at $\bar{r}_B$ from point source at $\bar{r}_A$	$p_{sc}(\bar{r}_B, \bar{r}_A)$	$p_{sc}^a(\bar{r}_B, \bar{r}_A) = p_{sc}(\bar{r}_A, \bar{r}_B)$	$p_{sc}^a(\bar{r}_B, \bar{r}_A) = p_{sc}(\bar{r}_A, \bar{r}_B)$

The complex problems involved in synthesizing transducer materials of prescribed electroacoustic properties is not the subject of the present paper. However, two groups of nonreciprocal transducers should be mentioned for which the creation of the adjoint transducers require, in principle, a relatively simple procedure. Transducers in both groups have symmetric conductivity, $\bar{\bar{\sigma}}$, permittivity $\bar{\bar{\epsilon}}$, permeability $\bar{\bar{\mu}}$, Hooke's tensor $\bar{\bar{c}}$; and magneto-electric tensors related by $\bar{\bar{\nu}} = -\bar{\bar{\tau}}_t$. Transducers of the first group have, in addition, the piezoelectric and piezomagnetic tensors related by

$$\bar{\bar{\alpha}} = \bar{\bar{\alpha}}_t'$$

$$\bar{\bar{\beta}} = -\bar{\bar{\beta}}_t'.$$

Transducers of the second group satisfy

$$\bar{\bar{\alpha}} = -\bar{\bar{\alpha}}_t'$$

$$\bar{\bar{\beta}} = \bar{\bar{\beta}}_t'.$$

Table 1 shows that the L adjoint to transducers in the first group can be realized by simply reversing the direction of the static magnetic induction $\bar{B}_0$ which biases the given transducer. (For

some gyrotropic media, e.g., ferrites, transposition of the permittivity and permeability tensors is also accomplished by the reversal of the magnetic induction. In fact, the reversal of $\bar{B}_0$ to form the L adjoint transducer is analagous to the reversal of the dc magnetic field necessary to obtain purely em reciprocity relationships for gyrotropic media [24].)

Table 1 also shows that the L^- adjoint to transducers of the second group can be realized simply by reversing the sign of the static charge distribution ρ_e^0, or equivalently, the static electric field and electric polarization which biases the transducer.

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The Diophantine Equation $p^2x^4 + 3px^2y^2 + y^4 = z^2$, *p an Odd Prime**

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(February 7, 1975)

The object of this paper is to prove:
THEOREM 1. *The equation*

$$p^2x^4 + 3px^2y^2 + y^4 = z^2$$

where p *is an odd prime, has no solutions in integers with* $xy \neq 0$ *if* $p = 5$ *or* $p \equiv 3, 7 \pmod{20}$ *or* $p \equiv 13 \pmod{40}$.

Key words: Diophantine equation, Infinite descent, Quadratic residues.

In this paper we indicate certain values of p, where p is an odd prime, such that the equation

$$p^2x^4 + 3px^2y^2 + y^4 = z^2 \tag{1}$$

has no integer solutions with $xy \neq 0$, and note some applications for the case $p=5$.

1. It is known that for $p = 1$ the equation (1) has no solution in integers with $xy \neq 0$ [1].[1] We now let p stand for an odd prime and consider the integer solutions of (1). We need only consider the solutions with $x \geq 0$, $y \geq 0$, $z \geq 0$ and $(x, y) = 1$. If $p \mid y$, then $p \mid z$, and dividing through by p^2 we get equation which is the same as (1) with x, y interchanged. Hence we assume that $p \nmid y$ (and so $y \neq 0$). Then it follows that x, y, z are prime each to each, and z is odd. Multiplying through by 4 we write (1) as

$$5(y^2)^2 + (2z)^2 = (2px^2 + 3y^2)^2$$

Since p is an odd prime we have $(y^2, 2z, 2px^2 + 3y^2) = 1$ or 2 according as y is odd or even. Hence [3]

$$y^2 = drs, 2z = \pm \frac{d}{2}(r^2 - 5s^2), 2px^2 + 3y^2 = \frac{d}{2}(r^2 + 5s^2) \tag{2}$$

where $d = 1$ or 4 according as r, s are integers of the same parity or of opposite parity, $(r, 5s) = 1$, $r \geq 1$, $s \geq 1$. From the first equation in (2) we have $r = R^2$, $s = S^2$, $y = \sqrt{d}\,RS > 0$, where R, S are integers, $(R, 5S) = 1$, $R \geq 1$, $S \geq 1$. Writing $D = \sqrt{d} > 0$, so that $D = 1$ or 2 according as

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[1] Figures in brackets indicate the literature references at the end of this paper.

R and S are of the same or of opposite parity we have from (2)

$$px^2 = \frac{D^2}{4}(R^2-5S^2)(R^2-S^2),\ y=DRS,\ z=\pm\frac{D^2}{4}(R^4-5S^4) \tag{3}$$

as the solution of (1) in px^2, y and z. If $R=S$, then $x=0$, $y^2=\pm s$. Suppose now that $R \neq S$, and consider the first equation in (3). It is clear that either $R > \sqrt{5}S$ or $R<S$; also that either $p \mid (R^2-S^2)$ or $p \mid (R^2-5S^2)$. We now prove

LEMMA 1. *Let* $p \mid (R^2-S^2)$. *Then*

(*i*) *if* $R > \sqrt{5}S$, *the first equation in* (3) *is impossible if* $p \equiv 3 \pmod 4$ *or* $p \equiv 5 \pmod 8$.

(ii) if $R<S$, there cannot occur a minimum value of $z>0$.

For, letting $\delta=(R-S,\ R+S)$ we have $\delta \leq 2$, and $\delta'=(R^2-5S^2,\ R^2-S^2)=(4S^2,\ R^2-S^2)=\delta^2$. Since $p \mid (R^2-S^2)$ we have $R \mp S \equiv 0 \pmod p$ and so from the first equation in (3) we have either

(a) $R \mp S = p\delta u^2, R \pm S = \delta v^2, R^2-5S^2 = \delta^2 w^2, (R > \sqrt{5}S)$

or

(b) $S \mp R = p\delta u^2, S \pm R = \delta v^2, 5S^2-R^2 = \delta^2 w^2, (R<S)$

where u, v, w are positive integers, relatively prime pairwise, $u \neq v$, $v \not\equiv 0 \pmod p$, and w is odd. The case (a) gives

$$3p\,u^2v^2 = p^2u^4 + v^4 + w^2$$

Hence $p \mid (v^4+w^2)$. This is impossible if $p \equiv 3 \pmod 4$. Also, if $p \equiv 5 \pmod 8$, the number $3pu^2v^2$ is not a sum of three squares, because this number is of the form $4^k(8N+7)$, $k \geq 0$. The result (i) of Lemma 1 follows.

The case (b) gives

$$p^2u^4+3pu^2v^2+v^4=w^2$$

which is the same as (1) and where

$$w^2=\frac{1}{\delta^2}(5S^2-R^2) \leq 5S^2-R^2=(\sqrt{5}S+R)(\sqrt{5}S-R) \tag{4}$$

Now, since $1 \leq R \leq S-1$, we have

$$\sqrt{5}-1<(\sqrt{5}-1)R^2+2R-1=\sqrt{5}R^2-(R-1)^2$$

$$\leq \sqrt{5}(S-1)^2-(R-1)^2=(\sqrt{5}S^2-R^2)-2(\sqrt{5}S-R)+(\sqrt{5}-1)$$

and so

$$\sqrt{5}S-R<\frac{1}{2}(\sqrt{5}S^2-R^2)$$

Also, plainly $\sqrt{5}S+R<\sqrt{5}S^2+R^2$ for all R, S under consideration. However, when R, S are both

odd, we have, since $2 \leq R+1 \leq S-1$,

$$\sqrt{5}+1 < \sqrt{5}(R+1)^2+(R-1)^2 \leq \sqrt{5}(S-1)^2+(R-1)^2$$
$$= (\sqrt{5}S^2+R^2)-2(\sqrt{5}S+R)+\sqrt{5}+1$$

and so
$$\sqrt{5}S+R < \frac{1}{2}(\sqrt{5}S^2+R^2)$$

Thus (with D defined above)
$$\sqrt{5}S \pm R < \frac{D}{2}(\sqrt{5}S^2 \pm R^2)$$

Hence the product of the two factors on the right hand side of (4) is

$$< \frac{D}{2}(\sqrt{5}S^2+R^2).\frac{D}{2}(\sqrt{5}S^2-R^2) = \frac{D^2}{4}(5S^4-R^4) \leq z^2, \text{ by (3).}$$

By the method of infinite descent the result (ii) of Lemma 1 follows.

LEMMA 2. *If* $p|(R^2-5S^2)$, *the first equation in* (3) *is impossible if* $p \equiv \pm 3, \pm 7 \pmod{20}$.

For, if $p|(R^2-5S^2)$, then $p \neq 5$, since otherwise $5|R$, contrary to the fact that $(R, 5S)=1$. Hence $\left(\frac{5}{p}\right)=1$. But $\left(\frac{5}{p}\right)=-1$ for $p \equiv \pm 3, \pm 7 \pmod{20}$, [2]. Lemma 2 follows.

From the Lemmas 1 and 2 now follows directly

THEOREM 1. *The equation* (1), *where* p *is an odd prime, has no solutions in integers with* $xy \neq 0$, *if*

$$p=5 \text{ or } p \equiv 3, 7 \pmod{20} \text{ or } p \equiv 13 \pmod{40}.$$

2. Applications for the case $p=5$.

THEOREM 2. *The equation*

$$x^4+3x^2y^2+y^4=5z^2 \tag{5}$$

has no solutions in integers with $x^2 \neq y^2$.

PROOF. We may suppose that $(x, y, z)=1$. Then x, y, z are prime each to each. Put $t=xy$ so that $(z, t)=1$. Then from the equation (5)

$$5z^2-t^2=(x^2+y^2)^2=U^2,\ 5z^2-5t^2=(x^2-y^2)^2=25V^2>0 \tag{6}$$

From the second equation in (6) we have, with $\partial=(z+t, z-t) \leq 2$,

$$z \mp t = 5\partial m^2,\ z \pm t = \partial n^2 \tag{7}$$

where m, n are relatively prime positive integers. From (7) and the first equation in (6) we then have

$$5^2m^4+3.5\,m^2n^2+n^4=(U/\partial)^2$$

which is impossible by theorem 1.

THEOREM 3. *The cubic curve*

$$\zeta^2+3\zeta+1=5\zeta\eta^2 \tag{8}$$

has only the (*finite*) *rational points* $(1, \pm 1)$.

For, let (ζ, η) be a finite rational point on (8). Clearly $\zeta\eta \neq 0$. Write (with l, m, z, n integers)

$$\zeta=\frac{l}{m},\ (l,\ m)=1,\ m>0,\ l\neq 0;\ \eta=\frac{z}{n},\ n>0,\ z\neq 0.$$

Then from (8) $$n^2(l^2+3lm+m^2)=5lmz^2$$

IF $5\nmid n$, we have, since $(l^2+3lm+m^2,\ lm)=1$,

$$l^2+3lm+m^2=5z^2,\ lm=n^2$$

Since $m>0$, we have $l>0$. Thus $l=x^2$, $m=y^2$, where x, y are integers, $(x,\ y)=1$, $xy\neq 0$, and so

$$x^4+3x^2y^2+y^4=5z^2$$

By theorem 2, this equation has only the solutions $x^2=y^2=1$. Hence $l=m=1$ and so $\zeta=1$, $\eta=\pm 1$.

For $n=5n'$ we have similarly $lm=5n'^2$, $l=5x^2$, $m=y^2$, leading to

$$5^2x^4+3.5x^2y^2+y^4=z^2,\ xy\neq 0$$

which is impossible theorem 1. Theorem 3 follows.

COROLLARY. *On writing* $\frac{1}{X}=\zeta$, $\frac{Y}{X}=\eta$, $(X\neq 0)$ *we see that the cubic curve*

$$X(X^2+3X+1)=5Y^2$$

has only the (finite) rational points $(0,\ 0)$, $(1,\ \pm 1)$.

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A Condition for the Diagonalizability of a Partitioned Matrix

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(March 5, 1975)

When U and V are diagonalizable matrices the diagonalizability of

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix}$$

is equivalent to the solvability in X of

$$UX - XV = N.$$

A corollary and simple generalization are given.

Key Words: Diagonalizable matrix; partitioned matrix

A square complex matrix A is termed "diagonalizable" if and only if A is similar to a diagonal matrix; that is, if and only if there exists a nonsingular matrix S such that $S^{-1}AS$ is diagonal. Our purpose is to prove the following necessary and sufficient condition for the diagonalizability of a partitioned matrix. We shall denote the class of n by n complex matrices by $M_n(C)$.

THEOREM 1. *Suppose* $\mathrm{U, V} \in \mathrm{M_n(C)}$ *are diagonalizable. Then*

$$\begin{bmatrix} \mathrm{U} & \mathrm{N} \\ 0 & \mathrm{V} \end{bmatrix} \quad \textit{is similar to} \quad \begin{bmatrix} \mathrm{U} & 0 \\ 0 & \mathrm{V} \end{bmatrix}$$

if and only if there is an $\mathrm{X} \in \mathrm{M_n(C)}$ *such that* $\mathrm{UX - XV = N}$.

PROOF: Suppose $S, T \in M_n(C)$ are invertible and are such that

$$SUS^{-1} = D \text{ and } T^{-1}VT = E$$

are diagonal. Then

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix} \quad \text{is similar to} \quad \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$$

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if and only if

$$\begin{bmatrix} D & SNT \\ 0 & E \end{bmatrix} \quad \text{is similar to} \quad \begin{bmatrix} D & 0 \\ 0 & E \end{bmatrix},$$

and

$$UX - XV = N$$

if and only if

$$D(SXT) - (SXT)E = SNT.$$

Thus, it suffices to assume from the outset that U and V are diagonal matrices.

Now suppose, first of all, that

$$U = \text{diag}\{u_1, \ldots, u_n\},$$

$$V = \text{diag}\{v_1, \ldots, v_n\},$$

and that

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix} \quad \text{is similar to} \quad \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix},$$

where $N = (n_{ij}) \epsilon M_n(C)$. We will show that if $u_i = v_j$, then $n_{ij} = 0$. It follows that if $X = (x_{ij})$ is defined by $x_{ij} = \dfrac{n_{ij}}{u_i - v_j}$ for $u_i \neq v_j$, and x_{ij} arbitrary otherwise, then $UX - XV = N$.

Denote by E_{ij} the n by n matrix all of whose entries are 0 except for a 1 in the i,j position. Then

$$\begin{bmatrix} I & tE_{ij} \\ 0 & I \end{bmatrix} \begin{bmatrix} U & N \\ 0 & V \end{bmatrix} \begin{bmatrix} I & -tE_{ij} \\ 0 & I \end{bmatrix} = \begin{bmatrix} U & N + t(v_j - u_i)E_{ij} \\ 0 & V \end{bmatrix}.$$

Because of this similarity we may assume without loss of generality that $n_{ij} = 0$ whenever $v_j \neq u_i$. If $N = 0$, we are finished. If not, we shall reach a contradiction. Suppose $N \neq 0$. Then via a permutational similarity we may assume that

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix}$$

is such that

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix},$$

$u_i = v_j = u$ for $i \leq k, j \leq \ell$,
$u_i \neq u$, $v_j \neq u$ for $i > k, j > \ell$, and
$N_1 \neq 0$ is k by ℓ. It is then clear that

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix}$$

is permutationally similar to

$$\left[\begin{array}{cc|cc} uI & N_1 & & \\ 0 & uI & 0 & \\ \hline & & * & N_2 \\ & 0 & 0 & * \end{array}\right]$$

Since

$$\begin{bmatrix} uI & N_1 \\ 0 & uI \end{bmatrix} \quad \text{is similar to} \quad \begin{bmatrix} uI & 0 \\ 0 & uI \end{bmatrix}$$

only if $N_1 = 0$, the assumption that $N \neq 0$ contradicts our original supposition that

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix} \quad \text{is similar to} \quad \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}.$$

Thus our original supposition does imply that $n_{ij} = 0$ whenever $v_j = u_i$ which in turn implies that $UX - XV$ is solvable.

Finally suppose on the other hand that $X \in M_n(C)$ is such that

$$UX - XV = N.$$

Then it is a simple computation that

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} U & N \\ 0 & V \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} U & -UX + N + XV \\ 0 & V \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix};$$

and the proof of the theorem is complete.

COROLLARY. *Suppose* $U, V \in M_n(C)$ *satisfy* $U^p = V^p = I$, $p \in I^+$. *Then*

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix}^p = I$$

if and only if there is an $X \in M_n(C)$ *such that* $UX - XV = N$.

PROOF: Since $U^p = V^p = I$, U and V are diagonalizable and the hypothesis of the theorem is satisfied. Now, if

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix}^p = I,$$

then $\begin{bmatrix} U & N \\ 0 & V \end{bmatrix}$ is diagonalizable

and thus similar to $\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$.

By the theorem, this implies $UX-XV=N$ is solvable. Conversely, if $UX-XV=N$, then

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix}^p \quad \text{is similar to} \quad \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}^p$$

which is equal to I. thus

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix}^p = I$$

which completes the proof of the corollary.

A straightforward generalization of Theorem 1 is as follows.

THEOREM 2. *Suppose each* $U_i \in M_n(C)$, $i=1, \ldots, k$, *is diagonalizable. Then*

$$\begin{bmatrix} U_1 & & N_{ij} \\ & \ddots & \\ 0 & & U_k \end{bmatrix} \quad \textit{is similar to} \quad \begin{bmatrix} U_1 & & 0 \\ & \ddots & \\ 0 & & U_k \end{bmatrix}$$

if and only if for each $i, j \leq k$ *there is an* $X \in M_n(C)$ (*depending on* i *and* j) *such that* $U_iX - XU_j = N_{ij}$.

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Linearly Independent Sets of Isotropic Cartesian Tensors of Ranks up to Eight

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This paper contains a complete listing of isotropic Cartesian tensors of ranks up to eight with their associated reduction equations for obtaining linearly independent sets whenever the reduction is called for. In particular, the listing is compiled only for isotropic tensors associated with the rotation group $O^+(3)$ of the three-dimensional underlying vector space. Based on an identity originally due to Capelli (1887), reduction equations for tensors of odd ranks beginning at rank five and even ranks beginning at rank eight are shown to be nontrivial. Significance of the computational result in both pure and applied mathematics is discussed.

Key words: Algebra; alternating tensor; Cartesian tensor; Capelli's identity; group representation; invariant; isotropic tensor; Kronecker delta; orthogonal group; tensor.

1. Introduction

With respect to some of the most important groups of linear transformations, Weyl [1][1] has written a definitive treatise on the algebraic problem of decomposing a space of tensors of a given rank into its irreducible invariant subspaces. In the course of proving his main results,

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[1] Figures in brackets indicate the literature references at the end of this paper.

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Weyl employed a powerful formal instrument known as Capelli's identity [1, p. 39]. Based on a theorem proved by Weyl [1, pp. 53-56], and by applying a special form of Capelli's identity, we first demonstrate a nontrivial problem. The problem is that of constructing explicit linearly independent sets of isotropic Cartesian tensors of odd ranks beginning at five and even ranks beginning at eight. Our algebraic results including all reduction equations are then given in full. An application of some of these results in mechanics of deformable media is described in a companion paper [2].

2. Statement of the Problem

To facilitate our application in mechanics, we adopt the three-dimensional Euclidean space as the underlying vector space in Cartesian coordinates. With respect to a fixed set of base vectors, each Cartesian tensor of rank n has 3^n real components denoted by $T_{ij\ldots(n \text{ times})}$, each subscript varying from 1 to 3. Let $A=[\alpha_{ij}]$ be a rotation matrix satisfying the conditions $\mathbf{A}^T\mathbf{A}=1$ and $\det \mathbf{A}=+1$. A Cartesian tensor of rank n, with respect to the three-dimensional proper orthogonal group $O^+(3)$, satisfies by definition the following transformation law:

$$T^{(\text{new})}_{jmq\ldots(n \text{ times})} = T^{(\text{old})}_{ikp\ldots(n \text{ times})}\,\alpha_{ij}\,\alpha_{km}\,\alpha_{pq}\ldots(n \text{ times}), \tag{2.1}$$

where repeated indices denote summation according to the usual convention with each index ranging from 1 to 3, and the α's are based on the rotation matrix with which new base vectors are expressed in terms of the old ones.

Within the context of this paper, we define a Cartesian tensor of rank n as isotropic if it satisfies the additional requirement that all components of the tensor remain constant for arbitrary rotations of the coordinate axes.[2] It is not difficult to show that there is no isotropic Cartesian tensor of rank 1; all isotropic Cartesian tensors of rank 2 are in the form of $\lambda\,\delta_{ij}$, λ being an arbitrary scalar; and all isotropic Cartesian tensors of rank 3 are scalar multiples of the alternating tensor ϵ_{ijk} which equals $+1$, 0, -1 for cyclic, acyclic, anti-cyclic permutations of the three indices i, j, k.[3] As proved in Weyl [1, pp. 53-56], every isotropic Cartesian tensor of even rank is expressible as a linear combination of products of the Kronecker deltas, δ_{ij}, δ_{km}, etc.,[4] and every isotropic Cartesian tensor of odd rank is given by a linear combination of terms formed of products of an appropriate number of Kronecker deltas with an alternating tensor. Such products of Kronecker deltas with or without the alternating tensor ϵ_{ijk} will be referred to as fundamental isotropic Cartesian tensors (abbrev. FICT). For every rank $n > 1$, the total number $N(n)$ of FICT's can be calculated from the following formulas:

For n even,
$$N(n) = \frac{n!}{\left(\frac{n}{2}\right)!\quad 2^{n/2}}. \tag{2.2}$$

For n odd,
$$N(n) = \frac{n!}{3!\quad \left(\frac{n-3}{2}\right)!\quad 2^{(n-3)/2}}. \tag{2.3}$$

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[2] Many writers prefer to use the term "hemitropic" or "gyrotropic" for tensors invariant under rotations of the proper orthogonal group.

[3] See, e.g., Jeffreys [3, pp. 66-68].

[4] The same result for even rank tensors was given apparently first by Franklin [4] in 1923.

Except for $n = 2,3,4$, and 6, Racah [5] showed that $N(n)$ in general exceeds $M(n)$, the total number of linearly independent FICT's. Using the theory of group representation of the three-dimensional rotation group, $O^+(3)$, he derived an explicit formula for the number $M(n)$ as follows:

$$M(n) = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \left[\binom{n}{2k} - \frac{1}{2}\binom{n}{2k-1}\right]\binom{2k}{k}. \tag{2.4}$$

The following table shows that for odd ranks beginning at 5 and even ranks beginning at 8, there exist linear combinations among FICT's which are identically zero:

TABLE I.

Rank n	2	3	4	5	6	7	8	9	10	11
Number of Distinct FICT's $N(n)$	1	1	3	10	15	105	105	1,260	945	17,325
Number of Linearly Independent FICT's $M(n)$	1	1	3	6	15	36	91	232	603	1,585

The problem now is to find these linear combinations, known as reduction equations, for ranks 5, 7, and 8, for use in explicit applications such as those described in [2].

3. Capelli's Identity

To obtain the reduction equations for isotropic tensors of odd ranks beginning at five and even ranks beginning at eight, we rely on special cases of a powerful formal instrument known as Capelli's identity.[5] In particular, we need only to apply the following two identities:

$$\begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix} = \epsilon_{ijk}\epsilon_{pqr}. \tag{3.1}$$

$$\begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} & \delta_{is} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} & \delta_{js} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} & \delta_{ks} \\ \delta_{mp} & \delta_{mq} & \delta_{mr} & \delta_{ms} \end{vmatrix} = 0. \tag{3.2}$$

[5] Weyl [1, pp. 39-47] introduced Capelli's identity to prove a fundamental theorem on orthogonal invariants [1, p. 53]. For our purposes here, Weyl's equation (9.1) in [1, p. 53] suffices as the basis for computation.

Identity (3.1) can be verified directly by expanding the 3×3 determinant. Identity (3.2) follows from the fact that there are only three possible values of the indices in a three-dimensional space, and since there are four columns on the left hand side, the indices of at least two columns must be equal.[6] Hence the determinant on the left of (3.2) must vanish. In light of (3.1), consider the determinant of (3.2) expanded in the minors of the first column:

$$\epsilon_{qrs}(\delta_{ip}\epsilon_{jkm} - \delta_{jp}\epsilon_{ikm} + \delta_{kp}\epsilon_{ijm} - \delta_{mp}\epsilon_{ijk}) = 0. \tag{3.3}$$

Since the first factor of this equation does not vanish identically, the linear combination of the following four isotropic tensors of rank five must:

$$\epsilon_{ijk}\delta_{mp} - \epsilon_{jkm}\delta_{ip} + \epsilon_{kmi}\delta_{jp} - \epsilon_{mij}\delta_{kp} = 0. \tag{3.4}$$

Equation (3.4) can be characterized by the index p, which is distinguished by occurring in each of the four Kronecker deltas. One can write down a total of five such equations, each characterized by one of the indices i, j, k, m, p. A glance at table 1 in the last section, however, reveals that for rank five, six out of ten distinct FICT's are required to form a linearly independent set. Hence one of those five equations based on (3.4) must be dependent. In section 4, we show how a linearly independent set of isotropic tensors of rank five can easily be obtained by using (3.4). A similar technique will be used in section 5 for rank 7 tensors, and in section 6 for rank 8 tensors. A vanishing linear combination such as eq (3.4) is usually referred to as a "null isotropic tensor."

4. Isotropic Tensors of Ranks 2, 3, 4, 5, and 6

For completeness, we list here distinct isotropic tensors of ranks 2, 3, 4, and 6, for which no reduction is needed.

TABLE 2.

Rank	Distinct and linearly independent fundamental isotropic tensors				
2	δ_{ij}				
3	ϵ_{ijk}				
4	$\delta_{ij}\delta_{km}$,	$\delta_{ik}\delta_{jm}$,	$\delta_{im}\delta_{jk}$		
6	$\delta_{ij}\delta_{km}\delta_{pq}$	$\delta_{ij}\delta_{kp}\delta_{mq}$	$\delta_{ij}\delta_{kq}\delta_{mp}$	$\delta_{ik}\delta_{jm}\delta_{pq}$	$\delta_{ik}\delta_{jp}\delta_{mq}$
	$\delta_{ik}\delta_{jq}\delta_{mp}$	$\delta_{im}\delta_{jk}\delta_{pq}$	$\delta_{im}\delta_{jp}\delta_{kq}$	$\delta_{im}\delta_{jq}\delta_{kp}$	$\delta_{ip}\delta_{jk}\delta_{mq}$
	$\delta_{ip}\delta_{jm}\delta_{kq}$	$\delta_{ip}\delta_{jq}\delta_{km}$	$\delta_{iq}\delta_{jk}\delta_{mp}$	$\delta_{iq}\delta_{jm}\delta_{kp}$	$\delta_{iq}\delta_{jp}\delta_{km}$

[6] Similar applications of (3.2) in mechanics are in the literature. See, e.g., Rivlin [6], Spencer and Rivlin [7], etc.

Independent sets of isotropic tensors of rank five have been studied by Cisotti [8] and Caldonazzo [9]. A particularly simple way of generating such a set is to write eq (3.4) in the following way:

$$\epsilon_{jkm}\,\delta_{ip} = \epsilon_{ijk}\,\delta_{mp} + \epsilon_{ikm}\,\delta_{jp} - \epsilon_{ijm}\,\delta_{kp}. \tag{4.1}$$

It is immediately clear from this equation, that any fundamental isotropic tensor of rank five in which the index i appears in the Kronecker delta can be expressed as a linear combination of fundamental tensors in which the index i appears in the alternator. A count of the number of the latter fundamental rank-five isotropic tensors reveals that there are six of them, viz:

$$\epsilon_{ijk}\,\delta_{mp},\quad \epsilon_{ijm}\,\delta_{kp},\quad \epsilon_{ijp}\,\delta_{km},\quad \epsilon_{ikm}\,\delta_{jp},\quad \epsilon_{ikp}\,\delta_{mj},\quad \epsilon_{imp}\,\delta_{jk}. \tag{4.2}$$

Since Weyl's result [1] implies that (3.4) or (4.1) exhaust all possible reduction equations for rank-five isotropic tensors, and since none of the six in (4.2) can be so reduced, we conclude that (4.2) is a linearly independent set.

5. Isotropic Tensors of Rank 7

The fundamental isotropic tensors of rank seven are products of an alternating tensor and two Kronecker deltas. Table 1 indicates that there are 105 distinct ones for that rank. By the procedure of the last section, it is a simple matter to eliminate all but 45 of these. Thus we need consider only the 45 distinct fundamental tensors generated from $\epsilon_{ijk}\,\delta_{mp}\,\delta_{qr}$ by those permutations of the indices which leave the subscript i with the alternator ϵ_{ijk}. Table 1 indicates that a further reduction is necessary since only 36 of these 45 fundamental tensors are linearly independent.

Null isotropic tensors in these 45 fundamental tensors can easily be constructed with the aid of eq (3.4). The product of eq (3.4) with δ_{qr} can be combined with eq (4.1) in which the indices q and r replace i and p, respectively, to give the following equation:

$$\begin{aligned}
&\epsilon_{ijk}\quad(\delta_{mp}\delta_{qr} - \delta_{qp}\delta_{mr}) + \epsilon_{ikm}\quad(\delta_{jp}\delta_{qr} - \delta_{qp}\delta_{jr})\\
+\ &\epsilon_{ijm}\quad(\delta_{qp}\delta_{kr} - \delta_{kp}\delta_{qr}) + \epsilon_{ikq}\quad(\delta_{mp}\delta_{jr} - \delta_{jp}\delta_{mr})\\
+\ &\epsilon_{imq}\quad(\delta_{jp}\delta_{kr} - \delta_{kp}\delta_{jr}) + \epsilon_{ijq}\quad(\delta_{kp}\delta_{mr} - \delta_{mp}\delta_{kr}) \;=\; 0\,.
\end{aligned} \tag{5.1}$$

This equation can be characterized by the two indices which appear in the product of Kronecker deltas of every term; i.e., p and r. Of the other equations which can be generated by permuting indices, 15 are distinct with a distinct pair of characteristic indices. These equations can be put in the form of a 45×15 matrix for reduction on a computer.

In order to simplify manipulation of symbols, it is useful to number the 45 fundamental isotropic tensors. This has been done in table 3, where the symbol $\langle n \rangle$ is assigned to the nth tensor of the table.

TABLE 3. *Coding of 45 isotropic tensors of rank 7, of which 36 are independent*

$\langle 0\rangle^* = \epsilon_{ijk}\delta_{mp}\delta_{qr}$	$\langle 1\rangle^* = \epsilon_{ijk}\delta_{mq}\delta_{pr}$	$\langle 2\rangle = \epsilon_{ijk}\delta_{mr}\delta_{pq}$
$\langle 3\rangle = \epsilon_{ijm}\delta_{kp}\delta_{qr}$	$\langle 4\rangle^* = \epsilon_{ijm}\delta_{kq}\delta_{pr}$	$\langle 5\rangle^* = \epsilon_{ijm}\delta_{kr}\delta_{pq}$
$\langle 6\rangle = \epsilon_{ijp}\delta_{km}\delta_{qr}$	$\langle 7\rangle^* = \epsilon_{ijp}\delta_{kq}\delta_{mr}$	$\langle 8\rangle = \epsilon_{ijp}\delta_{kr}\delta_{mq}$
$\langle 9\rangle = \epsilon_{ijq}\delta_{km}\delta_{pr}$	$\langle 10\rangle = \epsilon_{ijq}\delta_{kp}\delta_{mr}$	$\langle 11\rangle = \epsilon_{ijq}\delta_{kr}\delta_{mp}$
$\langle 12\rangle = \epsilon_{ijr}\delta_{km}\delta_{pq}$	$\langle 13\rangle = \epsilon_{ijr}\delta_{kp}\delta_{mq}$	$\langle 14\rangle = \epsilon_{ijr}\delta_{kq}\delta_{mp}$
$\langle 15\rangle^* = \epsilon_{ikm}\delta_{jp}\delta_{qr}$	$\langle 16\rangle^* = \epsilon_{ikm}\delta_{jq}\delta_{pr}$	$\langle 17\rangle = \delta_{ikm}\delta_{jr}\delta_{pq}$
$\langle 18\rangle = \epsilon_{ikp}\delta_{jm}\delta_{qr}$	$\langle 19\rangle^* = \epsilon_{ikp}\delta_{jq}\delta_{mr}$	$\langle 20\rangle = \epsilon_{ikp}\delta_{jr}\delta_{mq}$
$\langle 21\rangle = \epsilon_{ikq}\delta_{jm}\delta_{pr}$	$\langle 22\rangle = \epsilon_{ikq}\delta_{jp}\delta_{mr}$	$\langle 23\rangle = \epsilon_{ikq}\delta_{jr}\delta_{mp}$
$\langle 24\rangle = \epsilon_{ikr}\delta_{jm}\delta_{pq}$	$\langle 25\rangle = \epsilon_{ikr}\delta_{jp}\delta_{mq}$	$\langle 26\rangle = \epsilon_{ikr}\delta_{jq}\delta_{mp}$
$\langle 27\rangle = \epsilon_{imp}\delta_{jk}\delta_{qr}$	$\langle 28\rangle^* = \epsilon_{imp}\delta_{jq}\delta_{kr}$	$\langle 29\rangle = \epsilon_{imp}\delta_{jr}\delta_{kq}$
$\langle 30\rangle = \epsilon_{imq}\delta_{jk}\delta_{pr}$	$\langle 31\rangle = \epsilon_{imq}\delta_{jp}\delta_{kr}$	$\langle 32\rangle = \epsilon_{imq}\delta_{jr}\delta_{kp}$
$\langle 33\rangle = \epsilon_{imr}\delta_{jk}\delta_{pq}$	$\langle 34\rangle = \epsilon_{imr}\delta_{jp}\delta_{kq}$	$\langle 35\rangle = \epsilon_{imr}\delta_{jq}\delta_{kp}$
$\langle 36\rangle = \epsilon_{ipq}\delta_{jk}\delta_{mr}$	$\langle 37\rangle = \epsilon_{ipq}\delta_{jm}\delta_{kr}$	$\langle 38\rangle = \epsilon_{ipq}\delta_{jr}\delta_{km}$
$\langle 39\rangle = \epsilon_{ipr}\delta_{jk}\delta_{mq}$	$\langle 40\rangle = \epsilon_{ipr}\delta_{jm}\delta_{kq}$	$\langle 41\rangle = \epsilon_{ipr}\delta_{jq}\delta_{km}$
$\langle 42\rangle = \epsilon_{iqr}\delta_{jk}\delta_{mp}$	$\langle 43\rangle = \epsilon_{iqr}\delta_{jm}\delta_{kp}$	$\langle 44\rangle = \epsilon_{iqr}\delta_{jp}\delta_{km}$.

*Marks the nine tensors eliminated by our reduction scheme.

The numerical representation of tensors in table 3 enables us to present the nine reduction equations in a simpler fashion. For example, eq (5.1) becomes

$$\langle 0\rangle - \langle 2\rangle - \langle 3\rangle + \langle 5\rangle + \langle 10\rangle - \langle 11\rangle + \langle 15\rangle - \langle 17\rangle - \langle 22\rangle + \langle 23\rangle + \langle 31\rangle - \langle 32\rangle = 0. \tag{5.2}$$

The fifteen equations mentioned earlier can all be coded in a form similar to eq (5.2) in order to obtain a 45×15 matrix for computer solution. This matrix can be reduced by standard procedures to give nine reduction equations expressing nine tensors as linear combinations among the remaining 36. The nine reduction equations used to eliminate tensors $\langle 0\rangle$, $\langle 1\rangle$, $\langle 4\rangle$, $\langle 5\rangle$, $\langle 7\rangle$, $\langle 15\rangle$, $\langle 16\rangle$, $\langle 19\rangle$ and $\langle 28\rangle$ are shown in table 4, and the eliminated tensors are marked with asterisks in table 3. The remaining 36 fundamental tensors of rank 7 form a complete linearly independent set. Such a set is, of course, not unique and entails some arbitrariness in the choice of which tensors are eliminated.

TABLE 4. *Reduction equations for isotropic tensors of rank 7*

$$\langle 0\rangle = 2 + 6 - 8 - 10 + 11 - 12 + 13 - 18 + 20 + 22 - 23 + 24 - 25 - 37 + 38 - 43 + 44.$$

$$\langle 1\rangle = 2 + 9 - 10 - 12 + 13 - 21 + 22 + 24 - 25 - 43 + 44.$$

$$\langle 4\rangle = 3 - 6 + 8 + 9 - 11 - 13 + 14 + 27 - 29 - 30 + 32 + 37 - 38 + 39 - 40 - 42 + 43.$$

$$\langle 5\rangle = 3 - 6 + 8 + 12 - 13 + 27 - 29 - 31 + 32 - 33 + 34 + 37 - 38 + 39 - 40 + 43 - 44.$$

$$\langle 7\rangle = 8 + 10 - 11 - 13 + 14 - 36 + 37 + 39 - 40 - 42 + 43.$$

$$\langle 15\rangle = 17 + 18 - 20 - 24 + 25 - 27 + 29 + 33 - 34 - 39 + 40.$$

$$\langle 16\rangle = 17 + 21 - 23 - 24 + 26 - 30 + 32 + 33 - 35 - 42 + 43.$$

$$\langle 19\rangle = 20 + 22 - 23 - 25 + 26 - 36 + 38 + 39 - 41 - 42 + 44.$$

$$\langle 28\rangle = 29 + 31 - 32 - 34 + 35 - 37 + 38 + 40 - 41 - 43 + 44.$$

In all nine equations given in table 4, we have simplified further by dropping all brackets on the right hand side of each equation. The computational result is now complete for rank-seven isotropic tensors.

6. Isotropic Tensors of Rank 8

There are 105 fundamental tensors of rank 8, of which only 91 are linearly independent. The null isotropic tensors can be represented by determinants of matrices of Kronecker deltas as was done in eq (3.2). The number of distinct null vectors that can result from permutations of indices of eq (3.2) is 35, that is the number of ways the 8 indices can be divided up into two distinct groups of 4 each. These results can be put into a 105×35 matrix and reduced in the manner described in the previous section.

The reduction of this large matrix requires considerably more manipulation than does the 45×15 matrix of the last section. For this reason, it is convenient to have a method of coding the fundamental isotropic tensors by calculation rather than searching an arbitrary list. As it turns out, there is a simple way to do this. The letter indices of the Kronecker deltas are first interchanged with numbers, viz:

$$(i,\ j,\ k,\ m,\ p,\ q,\ r,\ s) \rightarrow (7,\ 6,\ 5,\ 4,\ 3,\ 2,\ 1,\ 0). \tag{6.1}$$

Each Kronecker delta then corresponds to a doublet of digits, and these are ordered according to the magnitude of the largest digit (Step 1). The smaller digits of each of the three largest doublets are then collected in order and arrayed from left to right (Step 2). Each of these digits is then reduced by one for each digit to the left of it that is smaller (Step 3). Finally the right hand digit is added to three times the middle digit and 15 times the left hand digit (Step 4). There results a unique number in the range 0 to 104 which characterizes the original fundamental isotropic tensor. The process is illustrated by the following example:

$$\begin{aligned}
\delta_{ir}\delta_{jq}\delta_{km}\delta_{ps} \quad \text{becomes} \quad &\text{(Step 1):} \quad (7,1)(6,2)(5,4)(3,0) \\
&\text{(Step 2):} \quad (7,1)(6,2)(5,4) \quad \text{or} \quad (1,2,4) \\
&\text{(Step 3):} \quad (1,\ 2-1,\ 4-1-1) \quad \text{or} \quad (1,1,2) \\
&\text{(Step 4):} \quad (1\times 15 + 1\times 3 + 2) \quad \text{or} \quad \langle 20\rangle
\end{aligned} \tag{6.2}$$

The inverse coding process is easily constructed. This coding procedure, though mysterious at first glance, relates in a straightforward way with the method of indexing the Kronecker deltas. It is a procedure which can be generalized to arbitrary even ranks of fundamental isotropic tensors. For the rank 8 case, it is convenient to have a dictionary of this coding to use in interpreting results. Table 5 is such a dictionary.

TABLE 5. *Dictionary of rank 8 isotropic tensors*

⟨0⟩ = *is jr kq mp*	⟨21⟩ = *ir jp ks mq*	⟨42⟩ = *iq jk ms pr*	⟨63⟩ = *im jr ks pq*	⟨84⟩ = *ik jp ms qr*
⟨1⟩ = *is jr kp mq*	⟨22⟩ = *ir jp jp ms*	⟨43⟩ = *iq jk mr ps*	⟨64⟩ = *im jr kq ps*	⟨85⟩ = *ik jp mr qs*
⟨2⟩ = *is jr km pq*	⟨23⟩ = *ir jp km qs*	⟨44⟩ = *iq jk mp rs*	⟨65⟩ = *im jr kp qs*	⟨86⟩ = *ik jp mq rs*
⟨3⟩ = *is jq kr mp*	⟨24⟩ = *ir jm ks pq*	⟨45⟩ = *ip js kr mq*	⟨66⟩ = *im jq ks pr*	⟨87⟩ = *ik jm ps qr*
⟨4⟩ = *is jq kp mr*	⟨25⟩ = *ir jm kq ps*	⟨46⟩ = *ip js kq mr*	⟨67⟩ = *im jq kr ps*	⟨88⟩ = *ik jm pr qs*
⟨5⟩ = *is jq km pr*	⟨26⟩ = *ir jm kp qs*	⟨47⟩ = *ip js km qr*	⟨68⟩ = *im jq kp rs*	⟨89⟩ = *ik jm pq rs*
⟨6⟩ = *is jp kr mq*	⟨27⟩ = *ir jk ms pq*	⟨48⟩ = *ip jr ks mq*	⟨69⟩ = *im jp ks qr*	⟨90⟩ = *ijks mr pq*
⟨7⟩ = *is jp kq mr*	⟨28⟩ = *ir jk mq ps*	⟨49⟩ = *ip jr kq ms*	⟨70⟩ = *im jp kr qs*	⟨91⟩ = *ij ks mq pr*
⟨8⟩ = *is jp km qr*	⟨29⟩ = *ir jk mp qs*	⟨50⟩ = *ip jr km qs*	⟨71⟩ = *im jp kq rs*	⟨92⟩ = *ij ks mp qr*
⟨9⟩ = *is jm kr pq*	⟨30⟩ = *iq js kr mp*	⟨51⟩ = *ip jq ks mr*	⟨72⟩ = *im jk ps qr*	⟨93⟩ = *ij kr ms pq*
⟨10⟩ = *is jm kq pr*	⟨31⟩ = *iq js kp mr*	⟨52⟩ = *ip jq kr ms*	⟨73⟩ = *im jk pr qs*	⟨94⟩ = *ij kr mq ps*
⟨11⟩ = *is jm kp qr*	⟨32⟩ = *iq js km pr*	⟨53⟩ = *ip jq km rs*	⟨74⟩ = *im jk pq rs*	⟨95⟩ = *ij kr mp qs*
⟨12⟩ = *is jk mr pq*	⟨33⟩ = *iq jr ks mp*	⟨54⟩ = *ip jm ks qr*	⟨75⟩ = *ik js mr pq*	⟨96⟩ = *ij kq ms pr*
⟨13⟩ = *is jk mq pr*	⟨34⟩ = *iq jr kp ms*	⟨55⟩ = *ip jm kr qs*	⟨76⟩ = *ik js mq pr*	⟨97⟩ = *ij kq mr ps*
⟨14⟩ = *is jk mp qr*	⟨35⟩ = *iq jr km ps*	⟨56⟩ = *ip jm kq rs*	⟨77⟩ = *ik js mp qr*	⟨98⟩ = *ij kq mp rs*
⟨15⟩ = *ir js kq mp*	⟨36⟩ = *iq jp ks mr*	⟨57⟩ = *ip jk ms qr*	⟨78⟩ = *ik jr ms pq*	⟨99⟩ = *ij kp ms qr*
⟨16⟩ = *ir js kp mq*	⟨37⟩ = *iq jp kr ms*	⟨58⟩ = *ip jk mr qs*	⟨79⟩ = *ik jr mq ps*	⟨100⟩ = *ij kp mr qs*
⟨17⟩ = *ir js km pq*	⟨38⟩ = *iq jp km rs*	⟨59⟩ = *ip jk mq rs*	⟨80⟩ = *ik jr mp qs*	⟨101⟩ = *ij kp mq rs*
⟨18⟩ = *ir jq ks mp*	⟨39⟩ = *iq jm ks pr*	⟨60⟩ = *im js kr pq*	⟨81⟩ = *ik jq ms pr*	⟨102⟩ = *ij km ps qr*
⟨19⟩ = *ir jq kp ms*	⟨40⟩ = *iq jm kr ps*	⟨61⟩ = *im js kq pr*	⟨82⟩ = *ik jq mr ps*	⟨103⟩ = *ij km pr qs*
⟨20⟩ = *ir jq km ps*	⟨41⟩ = *iq jm kp rs*	⟨62⟩ = *im js kp qr*	⟨83⟩ = *ik jq mp rs*	⟨104⟩ = *ij km pq rs*

The process of reducing the matrix by computer is facilitated by knowing something of the form of the results. Apparently, the reduction equations need never involve coefficients other than 1 and −1. Whether this is true in general remains, to the best of our knowledge, a conjecture. In light of this, the computer is instructed to abandon the test of any candidate for elimination whenever a coefficient other than ±1 appears in the reduced matrix.

The result of the reduction is the elimination of fundamental isotropic tensors ⟨0⟩, ⟨1⟩, ⟨12⟩, ⟨13⟩, ⟨34⟩, ⟨35⟩, ⟨44⟩, ⟨49⟩, ⟨58⟩, ⟨59⟩, ⟨68⟩, ⟨75⟩, ⟨76⟩, ⟨81⟩. The reduction equations are given in table 6. The remaining 91 tensors are a linearly independent set of isotropic tensors of rank 8.

TABLE 6. *Reduction equations for rank-eight isotropic tensors*

$$\langle 0 \rangle = 2+3-5-9+10+16-17-21+23+24-26-31+32+36-38-39+41-45+46+48-50-51+53+55-56+60-61-63+64+66-67-79+80+82-83-85+86+94-95-97+98+100-101.$$

$$\langle 1 \rangle = 2+6-8-9+11+16-17-21+23+24-26-45+47+48-50-54+55+60-62-63+65+69-70.$$

$$\langle 12 \rangle = -3+5+7-8+9-10+14+18-20-22+23-24+25+27-29-36+37+39-40-42+43-66+67+69-70-72+73+90-92-93+95+96-97+102-103.$$

$$\langle 13 \rangle = -3+5+6-8+14+18-20-21+23+28-29-66+67+69-70-72+73+91-92-94+95+102-103.$$

$$\langle 34 \rangle = 15-17-18+20+24-25-30+31+33-36+37-46+47+51-53-54+56+60-62-63+65+69-70+78-80-82+83-84+85+87-89-93+95+97-98+99-100-102+104.$$

$$\langle 35 \rangle = 15-16-18+20+21-23-25+26-30+31+33-36+38+40-41+45-46-48+50+51-53-55+56+79-80-82+83+85-86-94+95+97-98-100+101.$$

$$\langle 44 \rangle = -15+17+22-23-27+29+30-32-37+38+42-60+61+70-71-73+74+93-95-96+98+103-104.$$

$$\langle 49 \rangle = 4-5-7+8+10-11+16-17-19+20-21+22+24-25-31+32+36-38-39+41-45+46+48-51+52+60-61-63+64+66-67+78-79-84+86+87-89-93+94+99-101-102+104.$$

$$\langle 58 \rangle = -4+5+7-8-10+11+19-20-22+23+25-26-36+37+39-40-42+43+51-52-54+55+57-66+67+69-70-72+73+96-97-99+100+102-103.$$

$$\langle 59 \rangle = -4+5+7-8-10+11-16+17+19-20-27+28+31-32-36+38+39-41+45-46+51-52-54+56+57-60+61-66+67+69-71-72+74+93-94-99+101+102-104.$$

$$\langle 68 \rangle = 4-5-7+8+10-11-31+32+36-38-39+41+46-47-51+53+54-56-61+62+66-69+71.$$

$$\langle 75 \rangle = -15+17+18-20-24+25+46-47-51+53+54-56+77+82-83-87+89+90-92-97+98+102-104.$$

$$\langle 76 \rangle = -15+16+18-20-21+23+25-26-31+32+36-38-39+41+46-47-51+53+54-56+77+82-83-85+86-87+88+91-92-97+98+100-101+102-103.$$

$$\langle 81 \rangle = -4+5+7-8-10+11+19-20-22+23+25-26+82+84-85-87+88+96-97-99+100+102-103.$$

In all 14 equations given in table 6, we have also simplified further by dropping all brackets on the right hand side of each equation. The computational result is now complete for rank-eight isotropic tensors.

7. Significance of Computational Results

The computational results presented here were motivated by a direct application in mechanics as described in a companion paper [2]. It is probable, however, that further applications in theoretical physics will result in view of the close connection between isotropic tensors and the decomposition of a tensor space.[7] Since all laws in physics are necessarily tensorial equations, the tool we have developed here would be particularly useful in reducing complicated physical laws into uncoupled irreducible equations.

Two additional comments may be of interest to our readers. On the abstract level, our results may be used to generate examples of mathematically interesting algebras. Tensors invariant to transformations by elements of subgroups of the orthogonal group can be developed also. Such tensors are sometimes called "anisotropic tensors,"[8] and complete sets of them must contain the isotropic tensors developed in this paper.[9] At the practical level, our results can be used to determine whether two isotropic tensors of equal rank are indeed the same, since the question of "sameness" for two isotropic tensors of equal rank is best settled by expressing them in terms of a linearly independent set.[10]

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[7] See, e.g., Fano and Racah [10], Coope et. al [11, 12].

[8] See, e.g., Pipkin and Rivlin [13].

[9] A convenient listing for the crystal symmetry groups on such invariant tensors up to rank four is given by Lokhin and Sedov [14]. Additional results related to isotropic and anisotropic tensors may also be found in a paper by Smith [15].

[10] This question cannot be answered without the table of reduction equations as given in this paper for ranks up to eight. Smith [16] has shown that, for rank-eight tensors, a set of 91 linearly independent elements can be constructed via the "tableau" method. His results offer an alternate approach to the same problem, but contained no explicit reduction equations.

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Maximal Network Flows, Matroids and Matchings

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This paper represents new proofs for some known network flow results. First, maximal network flows are expressed as matroids, and then Gale's characterization of matroids is used to prove the existence of various lexicographic network flows. Second, an independence property of maximum flows is proved and this property is related to an independence property of Brualdi and Scrimger.

Key words: Independence properties; matroids; maximal flows; network flows.

1. Network Flows as Matroids

This section shows how a network flow may be regarded as a matroid and then shows how Gale's characterization of matroids [8][1] can be used to prove the existence of some lexicographic network flows [7,9].

A matroid on a finite set E can be defined [3] as a family F of subsets of set E with the property that

(1) if $A' \subseteq A$ and $A \in F$, then $A' \in F$, and

(2) for any subset $E' \subseteq E$, all maximal members of F that are contained in E' have the same cardinality.

This definition is the most appropriate for the purpose of this paper. A survey of other definitions appears in [2].

Let $N = (X,A)$ be a finite, directed network with node set X and arc set A. Let $S \subseteq X$ denote the source nodes, and let $T \subseteq X$ denote the sink nodes, where $S \cap T = \phi$. Our interest is in integer valued flows[2] that are feasible relative to the given integer arc capacities. It is technically convenient to assume that each arc in A has a capacity equal to one and to compensate by assuming that more than one arc may exist between two nodes. (The reader is asked to make the required translations.) Note that with these conventions each flow corresponds to a unique set of arcs (those that carry flow) so that one can speak of maximal flows.

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[1] Figures in brackets indicate the literature references at the end of this paper.

[2] Once the relationships between networks with integer flows and matroids are understood, the theory can be extended to polymatroids [4] and networks with noninteger capacities and flows. A polymatroid P is a compact nonempty subset of $R_+^{|E|}$ (the positive part of real $|E|$ space) such that

(a) $0 \leq x^0 \leq x' \in P \Rightarrow x^0 \in P$,

(b) for every $a \in R_+^{|E|}$ every maximal $x \in P$ such that $x \leq a$ has the same sum Σ_{x_j}.

Here maximal x means there is no $x \leq x$, $x' \neq x$, having the properties of x, and $a \leq b$ means $a_j \leq b_j$ for all j.

Let E_S be the set of all arcs of the form (x,y), $x \in S$, and let E_T be the set of all arcs of the form (x,y), $y \in T$. Let F_S be the set of all subsets of E_S that carry flow in some feasible integer flow in network N. Similarly, let F_T be the set of all subsets of E_T that carry flow in some feasible integer flow in network N.

THEOREM 1: *The set* F_S *is a matroid on set* E_S.

PROOF: To see that condition (1) in the definition of a matroid is satisfied, suppose there is a feasible flow in network N that uses to capacity arcs $A \subseteq E_S$. Then for any subset $A' \subseteq A$, we can construct a feasible flow simply by deleting the units of flow emanating from the arcs in $A - A'$. Hence, condition (1) is satisfied.

Let E'_S be any subset of E_S. Consider all the maximal members of the set of integer feasible flows in network N that use only source arcs in set E'_S. Suppose not all these maximal flows have the same value. Then, disregard the arcs in set $E_S - E'_S$, and try to increase the smallest valued maximal flow in the network by using Ford and Fulkerson's flow augmenting algorithm [6], p. 17. It is known that this algorithm must increase the flow on this restricted network (arcs $E_S - E'_S$ are avoided) by an integer amount since the current flow is not maximum on this restricted network. Furthermore, the algorithm does not decrease the flow on any arc adjacent to a source node. Thus all maximal integer flows that use set E'_S have the same cardinality. Q.E.D.

THEOREM 2: *The set* F_T *is a matroid on set* E_T.

The proof is similar to the proof of Theorem 1.

Let F be any set of subsets of a totally ordered set E. A member $A \in F$ is said to be *optimal* if for every set $B \in F$, there exists a univalent mapping f from B to A such that $f(b) \geq b$ for all b in B. Note that set A must be a maximum cardinality member of F.

For example, if there is a preference on the order in which the source arcs are to be used, then an optimal flow is a flow that uses the source arcs in the order of greatest preference, i.e., the most preferred source arc if possible, then the second most preferred source arc if possible, etc.

Clearly, not all sets of subsets have an optimal member. For example, a student has a choice between a grade of A in a single course or a grade of B in each of two courses. The set $\{A,BB\}$ possesses no optimal member for the total ordering $A > B$.

THEOREM 3: (Gale) *If* F *is a matroid, then* F *has an optimal member.*

The proof of this theorem can be found in [8], p. 178.

THEOREM 4: *If the source arcs in a network are ordered, then there exists a maximum flow that uses the source arcs optimally.*

The proof is immediate from Theorems 1 and 3. A constructive proof for Theorem 4 that uses only network flow arguments appears in [9].

A dynamic network is a network whose arcs not only have a capacity but also a traverse time, and flow units travel through time in a dynamic network. An *earliest departure dynamic flow* for p time periods is defined as a flow from the sources to the sinks in a dynamic network in which V_i is maximised for each $i = 0, 1,\ldots,p$, where V_i denotes the number of flow units that can leave the sources in the first i time periods. A *latest departure dynamic flow* for p time periods can be defined as a flow from the sources to the sinks in a dynamic network in which v_i is maximised for $i = 0, 1, \ldots,p$ where v_i denotes the number of flow units leaving the sources during the last i time periods, i.e., from time $p - i$ to time p. Similarly, an earliest arrival dynamic flow and a latest arrival dynamic flow can be defined. By definition, each of these four types of flows is a maximum flow.

A dynamic network flow for p time periods is equivalent to a flow in a time-expanded, non-dynamic replica $N(p)$ of the original dynamic network N. In the time expanded network $N(p)$, each node x in network N is represented by $p + 1$ nodes $x(0),x(1),\ldots,x(p)$, that represent node x at each time period. The arcs are replicated similarly. See [6], p. 144, for a detailed description of how network $N(p)$ can be constructed from network N. Theorem 4 can be applied to network $N(p)$ to show the existence of a maximum flow that uses the source (sink) arcs optimally.

THEOREM 5: *There exists an earliest departure (arrival) dynamic flow in network* N *for* p *time periods, for* p = 0,1,....

PROOF: Form an ordering of all the source (sink) arcs in network $N(p)$ as follows: all source (sink) arcs originating from a source (sink) node at time i are in the ith place in the order. By Theorem 4, there exists an optimal, maximal flow on network $N(p)$, and this flow corresponds to an earliest departure (arrival) dynamic flow in the network N. Q.E.D.

A proof for the existence of this flow using network flow techniques first appeared in [7].

THEOREM 6: *There exists a latest departure (arrival) dynamic flow in network* N.

The proof is similar to the proof for Theorem 5. A proof for this theorem that uses only network flow arguments can be found in [9].

2. An Independence Property of Network Flows

This section presents another proof for the independence property of source arcs and sink arcs [9] and shows how this property is related to an independence theorem of Brualdi and Scrimger [1].

THEOREM 7: (Independence of Source Arcs and Sink Arcs).

Let ϕ *and* θ *be any two maximal integer flows on a network* N = (X,A). *Let* $S_\phi \subseteq E_S$ *and* $S_\theta \subseteq E_S$ *denote the source arcs used by flows* ϕ *and* θ *respectively. Similarly, let* $T_\phi \subseteq E_T$ *and* $T_\theta \subseteq E_T$ *denote the sink arcs used by flows* ϕ *and* θ *respectively. Then, there exists a flow* γ *such that* $S_\gamma = S_\phi$ *and* $T_\gamma = T_\theta$.

PROOF: Let $Z \subseteq A$ denote any minimum cut separating the sources from the sinks in network N. We know from the Max Flow-Min Cut Theorem [6], p. 11, that all maximal flows saturate the arcs of cut Z. Hence maximal flows ϕ and θ saturate cut Z.

Now construct a flow that is identical to flow ϕ on all the arcs on the "source side" of cut Z. This flow, of course, saturates cut Z. Then let the flow proceed from cut Z to the sinks along the arcs chosen by flow θ. Clearly, such a construction is possible since cut Z is saturated by flows ϕ and θ. The resulting flow satisfies the conditions of the theorem. Q.E.D.

Theorem 7 is called an independence theorem for maximal flows because it demonstrates the independence between the source arcs used by the flow and the sink arcs used by the flow. A constructive proof for Theorem 7 first appeared in [9].

A similar independence theorem appears in [1], p. 253, for matchings. We shall consider only the finite case of this theorem: given two finite sets S and T, associate with each $s \in S$ a subset $r(s) \subseteq T$. A matching is defined as any univalent mapping ϕ of a subset S_ϕ of S into T such that if $s \in S_\phi$, then $\phi(s) \subseteq r(s)$. A matching ϕ is said to be maximal if there is no other matching θ such that $S_\phi \subset S_\theta$. Define $T_\phi = \phi(S_\phi)$.

THEOREM 8: (Brualdi and Scrimger). *Let* ϕ *and* θ *be any two maximal matchings between finite sets* S *and* T. *Then there exists a maximal matching* γ *such that* $S_\gamma = S_\phi$ *and* $T_\gamma = T_\theta$.

The proof in [1] uses set theoretic arguments. For the finite case, Theorem 8 is clearly a special case of Theorem 7 for a network N with only source nodes and sink nodes.

Other examples of matching theory results proved by network flow methods can be found in [5].

For dynamic flows, Theorem 7, together with Theorems 5 and 6, asserts the existence of a maximal dynamic flow that has both a latest departure schedule and an earliest arrival schedule and also the existence of a maximal dynamic flow that has both an earliest departure schedule and a latest arrival schedule.

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A Note on the Metrizability of Spaces With Countably Based Closed Sets

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The main result of this note is a generalization of an earlier theorem on the metrizability of spaces with countably based closed sets. Use is made of some results related to co-convergent spaces which are spaces having countably based compact sets.

Key words: Co-convergent; contra-convergent; Nagata spaces; open neighborhood assignments; stratifiable spaces; U-linked sequences.

C. E. Aull in [1][1] defines a D_1-*space* as a topological space in which every closed set F has a countable local base $\{B_n(F)\}$: i.e., for each $n \in N$, the set of positive integers, $B_n(F)$ is an open set containing F; and if $F \subset R$, where R is open, then $B_k(F) \subset R$ for some $k \in N$.

An *open neighborhood assignment* (ONA) is defined in [2] as a function

$$U : X \times N \to \cup \{N(x) : x \in X\}$$

such that $x \in U(x,n) \equiv U_n(x)$ where X is a topological space and $N(x)$ is the collection of open neighborhoods of x. If U is an ONA then the sequence $\{y_n\}$ is *U-linked* to $\{x_n\}$ if $y_n \in U_n(x_n)$ for all n. Using the notation $Cp\{x_n\}$ for the set of cluster points of $\{x_n\}$ we define a space to be *co-convergent* (*contra-convergent*) if $Cp\{x_n\} \subset Cp\{y_n\}$ ($Cp\{y_n\} \subset Cp\{x_n\}$) whenever $\{y_n\}$ is U-linked to $\{x_n\}$. If on X there is an ONA U having some property P we shall say "X is P" or "U is P." Finally for any $S \subset X$ and ONA U we have $U_n(S) \equiv \cup \{U_n(x) : x \in S\}$.

The following two theorems were proved in [2]:

THEOREM 1: X *is metrizable iff it is a co-convergent, contra-convergent* T_0*-space.*

THEOREM 2: *The following are equivalent on a space* X:

(a) X *is co-convergent.*

(b) There exists an ONA U *on* X *such that for any countably compact* K *and open* R *containing* K, $U_n(K) \subset R$ *for some* $n \in N$.

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[1] Figures in brackets indicate the literature references at the end of this paper.

(c) There exists an ONA U *on* X *such that for any convergent sequence* $\{x_n\}$ *with limit* x_0 *and open* R *containing* $\{x_k : k = 0,1,\ldots\}$, $U_n(\{x_k : k = 0,1,\ldots\}) \subset R$ *for some* $n \in N$.

A characterization of a *natural-D_1* space is as a space X on which there is an ONA U such that for every closed set F, $\{U_n(F)\}$ is a local base for F. We note that if X is T_2 and U is a natural-D_1 ONA then it is co-convergent.

We have an analogue of Theorem 2:

THEOREM 3: *Let* U *be an ONA on* X. *Then the following are equivalent:*

(a) U *is natural-*D_1.

(b) For any sequence $\{x_k\}$, $\{U_n(\Delta\{x_k\})$ *is a local base for* $\Delta\{x_k\}$ *where* $\Delta\{x_k\} \equiv Cl\{x_n : n = 1,2,\ldots\}$.

PROOF: Only (b) implies (a) requires any consideration. Suppose F is closed and is contained in an open R. If for each n there is a $z_n \in U_n(F) - R$ then there is a sequence $\{x_n\}$ in F with $\{z_n\}$ U-linked to $\{x_n\}$. But then $\Delta\{x_n\} \subset F$ and for some k, $U_k(\Delta\{x_n\}) \subset R$ implying the contradiction $z_k \notin R$.

The following result was proved in [3]:

THEOREM 4: *If* X *has at most a finite number of isolated points it is compact and metrizable iff it is natural-*D_1 *and Hausdorff.*

We shall generalize Theorem 4, by using Aull's result in [1] that every regular, D_1-space is the union of a countably compact set and a set of isolated points.

THEOREM 5: X *is compact and metrizable iff it is natural-*D_1 *and Hausdorff,*

PROOF: We need only consider the sufficiency part of the proof, the necessary part being the same as for Theorem 4.

Let U be a natural-D_1 ONA on X. Without loss of generality we can assume U is nested. Furthermore $\{U_n(x)\}$ is a local base for each $x \in X$. X is regular, for if F is closed and $x \in F$ and if for all n there is a $z_n \in U_n(x) - U_n(F)$, then there is a sequence $\{y_n\}$ in F such that $\{z_n\}$ is U-linked to $\{y_n\}$. It follows that $\{z_n\}$ converges to x. Hence there is an $M > 0$ such that for all $k > M$, $z_k \in X - F$. Since U is natural-D_1 there is an $n > M$ such that $U_n(F) \subset X - \Delta\{z_k\}_{k=M}^{\infty}$, which implies $U_n(y_n) \subset X - \Delta\{z_k\}_{k=M}^{\infty}$, contradicting $z_n \in U_n(y_n)$.

Let $X = C \cup I$ where C is countably compact and I is a set of isolated points of X. We can assume $C \cap I = 0$. Again without loss of generality we can let $U_n(x) = \{x\}$ for all $x \in I$ and all n. Let $\{y_n\}$ be U-linked to $\{x_n\}$ and $y \in Cp\{y_n\}$. Then there is a subsequence $\{y_{n_k}\}$ *of* $\{y_n\}$ converging to y. If $x_{n_k} \in I$ for infinitely many k, then $\{x_{n_k}\}$ clusters at y. If $\{x_{n_k}\} \in C$ for infinitely many k, $\{x_{n_k}\}$ clusters at some $x \in C$, implying by the co-convergence of U that $x = y$. Hence U is contra-convergent and by Theorem 1, X is metrizable.

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Charge Profiles in Parallel-Plate Ionization Chambers

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Numerical integration of the differential equations describing charge transport in plane-parallel ionization chambers allows to obtain charge profiles in addition to current-voltage curves. It is found that charge densities may exhibit maxima within a certain range of saturation. The effect is interpreted in terms of field modulation of carrier velocities.

Key words: Charge profiles; ionization chambers; numerical integration.

1. Introduction

The theory of the ionization chamber has been discussed in numerous papers since the turn of the century. Approximate solutions were given by Mie [1][1] and Boag and Wilson [2]. The literature has been surveyed by Greening [3].

With the advent of modern computational methods precise numerical solutions can be obtained of many of the differential equations which until now defied attempts to obtain such solutions.

By these methods, Armstrong-Tate [4] have solved the problem of the charge transport in parallel-plate ionization chamber. These authors calculated current-voltage curves and field profiles and compared their computer solution with the approximation given by previous authors. In this paper we use a computer method to calculate charge profiles. Results indicate novel features which so far have not been commented on. Our method of computation is similar to that of Armstrong-Tate. The use of different variables leads, however, to more advantageous conditions. Therefore, we have included a summary description of the calculus.

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[1] Figures in brackets indicate the literature references at the end of this paper.

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2. Basic Equations

We limit ourselves to steady-state conditions. The terms containing time disappear and one obtains the following well-known system of equations (for definition of symbols see table 1).

$$\begin{aligned} \epsilon dE'/dx' &= \rho'_+ - \rho'_- \\ di'_+/dx' &= -(\alpha'/e)\rho'_+ \rho'_- + q \\ -di'_-/dx' &= -(\alpha'/e)\rho'_- \rho'_+ + q \\ i'_+ &= \mu'_+ \rho'_+ E' \\ i'_- &= \mu'_- \rho'_- E' \\ E' &= -dV'/dx' \end{aligned} \tag{1}$$

These equations are completed by the usual boundary conditions:

$$\begin{aligned} \left(\int_0^d\right) E'(x')dx' &= V'_0 \\ \rho'_+(0) &= \rho'_-(d) = 0 \end{aligned} \tag{2}$$

By the introduction of the reduced, nondimensional quantities, (also defined in table 1) eqs (1) and (2) are transformed into eqs (3) and (4) which contain only 2 numerical parameters. Our equations are different from those of reference [4] because we use different reduction factors.

$$\begin{aligned} dE/dx &= \rho_+ - \rho_- \\ di_+/dx &= -\alpha\rho_+ \rho_- + 1 \\ -di_-/dx &= -\alpha\rho_+ \rho_- + 1 \\ i_+ &= \rho_+ E \\ i_- &= \kappa\rho_- E \\ E &= -dV/dx \end{aligned} \tag{3}$$

and

$$\begin{aligned} \left(\int_0^d\right) E dx &= V_0 \\ \rho_+(0) &= \rho_-(1) = 0 \end{aligned} \tag{4}$$

The eqs (3) may be transformed into

$$d\rho_+/dV = [\rho_+ (\rho_+ - \rho_-) + \alpha\rho_+ \rho_- - 1]/E^2$$
$$dE/dV = -(\rho_+ - \rho_-)/E \tag{5}$$

Introducing as a new variable $u = \rho_+ - \rho_-$, one obtains

$$\rho_- = [(i/E) - u]/(1 + \kappa)$$
$$\rho_+ = [u + i/\kappa E]\kappa/(1 + \kappa) \tag{6}$$

$$\frac{du}{dx} = \frac{\kappa + 1}{\kappa}\left[\frac{d\rho_+}{dx} + \frac{iu}{\kappa E^2}\right] \tag{7}$$

These relations can be simplified furthermore by introducing the variables $E^* = E/i$ and $V^* = V/i^2$. Substitution of the eqs (6) and (7) into eq (5) allows to eliminate i and gives

$$\frac{du}{dV^*} = -\frac{\kappa+1}{\kappa E^{*2}}\left[\frac{\kappa u^2}{(1+\kappa)^2}(\alpha-\kappa-1) - \frac{u\alpha(\kappa-1)}{(\kappa+1)^2 E^*} - \frac{\alpha}{(\kappa+1)^2 E^{*2}} + 1\right]$$
$$dE^*/dV^* = -u/E^* \tag{8}$$

3. Numerical Integration

The variable u must have a zero at a point $0 < x_0 < 1$; thus $u(x_0) = 0$. Since $\rho_-(1) = 0$, u increases monotonously from $u = 0$ at $x = x_0$ to $u = \rho_+$ at $x = 1$. Thus $i = \rho_+(1)E(1) = u(1)E(1)$. With $E^* = E/i$ this gives $u(1)E^*(1) = 1$.

Equations (8) are now integrated for increasing values of u from $u = 0$ to a maximum value which satisfies the relation $uE^* = 1$ and corresponds to $x = 1$. Between $x = x_0$ and $x = 0$, u decreases from 0 to a minimum value $u(0) = -\rho_-(0)$. The same argument used above gives $\kappa\rho_-(0)E(0) = i$ or $-\kappa u(0)E^*(0) = 1$. The integration terminates when the product uE^* satisfies the relation $-uE^* = 1/\kappa$ which corresponds to $x = 0$.

The value of the potential V is defined by postulating $V^*(x_0) = 0$.

The integration is performed in terms of the positive parameter $E^*(x_0) = E^*_m$. For $E \to 0$ (linear region) one has $\rho_+ = \rho_-$ and $di/dx = 0$. This gives $\rho_+\rho_- = 1/\alpha$ and $i = (\rho_+ + \kappa\rho_-)E = (1 + \kappa)E/\alpha^{1/2}$ or $E^* = \alpha^{1/2}/(1 + \kappa)$. This is the lower limit of E^*_m.

This initial condition in terms of a known quantity assures that the system (8) always has a solution. Armstrong-Tate [4] use an initial condition which contains an assumed value of the field strength at one electrode. The equations of the problem admit however a solution only for values of the field higher than a certain minimum value. Since this is not known in advance, one is not sure to obtain a solution.

Therefore we have the boundary conditions

$$
\begin{gathered}
u(x_0) = 0 \\
0 < x \leqslant x_0 : -(1/\kappa) \leqslant uE^* < 0 \\
x_0 < x \leqslant 1 : 0 < uE^* \leqslant 1 \\
E^*(x_0) = E^*_m,\ \alpha^{1/2}/(1+\kappa) < E^*_m \\
V^*(x_0) = 0
\end{gathered}
\tag{9}
$$

The current is obtained from the relation $dV/dx = -E$, giving

$$
dx = -i\, dV^*/E^* \tag{10}
$$

Integration over x for $0 < x < 1$ gives

$$
i = -1/\int_{V^*(0)}^{V^*(1)} dV^*/E \tag{11}
$$

Since $V^* = V/i^2$, the applied voltage $V_0 = V(0) - V(1)$ is given by

$$
V_0 = i^2\,[V^*(0) - V^*(1)] \tag{12}
$$

$V^*(0)$ and $V^*(1)$ are, respectively, the values of V^* which are obtained when u has reached its extreme values given by the eqs (9).

4. Current–Voltage Curves

For air at NTP, Scott and Greening [5] have given $\mu'_+ = 1.59 \times 10^{-4}$ m$^2/V$ sec, $\mu'_- = 2.21 \times 10^{-4}$ m$^2/V$ sec, and a ratio "recombination coefficient to elementary charge" = 4790 cm^3/esu sec which in our notation give a reduced recombination coefficient $\alpha = 0.799$ and $\kappa = 1.39$. We have carried out the integration with these values. Results are represented in figure 1, curve A, which gives i/i_s versus $(i/i_s)/V^2$.

In recent measurements Ritz and Attix [6] found a value $\mu' = 1.89 \times 10^{-4}$ m$^2/V$ sec for the mobilities of both positive and negative ions; their value of the ratio "recombination coefficient to elementary charge" was 4717 cm^3/sec esu which in our notation gives $\alpha = 0.662$. When the carrier mobilities are equal one has $\kappa = 1$ and eqs (8) are considerably simplified, containing then only the single material parameter α. In addition the system is symmetrical with regard to the plane $x = 1/2$. Boundary conditions are also simplified. Figure 1, curve B, gives the computer solution for this case. In the saturation region curves A and B coincide. Even beyond this region, differences remain small. We have also carried out a series of integration for $\kappa = 1$ and values of α between 0.1 and 0.9. Results are given in figure 2.

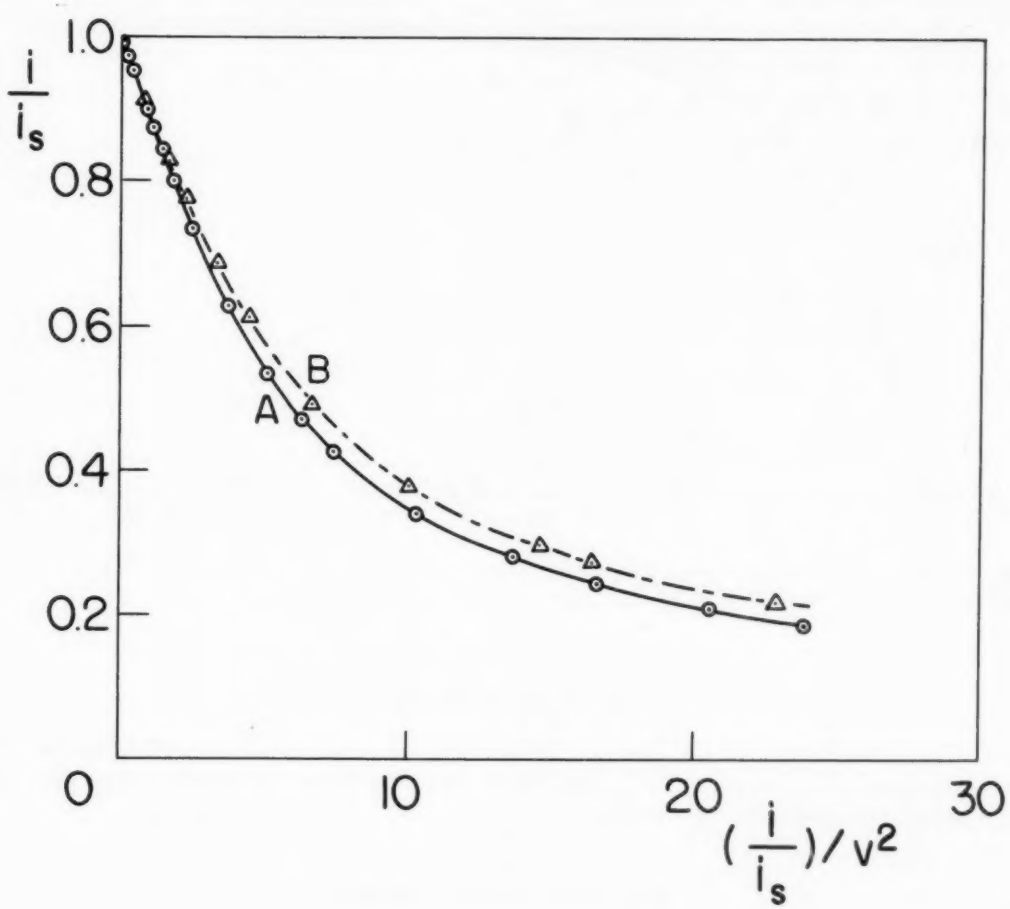

FIGURE 1. *Saturation curves: A) computer solution for* $\mu'_+ = 1.59\times10^{-4}\,m^2/V\,sec$; $\mu'_- = 2.21\times10^{-4}\,m^2/V\,sec$; $\alpha = 0.799$. *B) Computer solution for* $\alpha = 0.662$ *and* $\mu'_+ = \mu'_- = 1.89\times10^{-4}\,m^2/V\,sec$. *Circles and triangles indicate values actually calculated.*

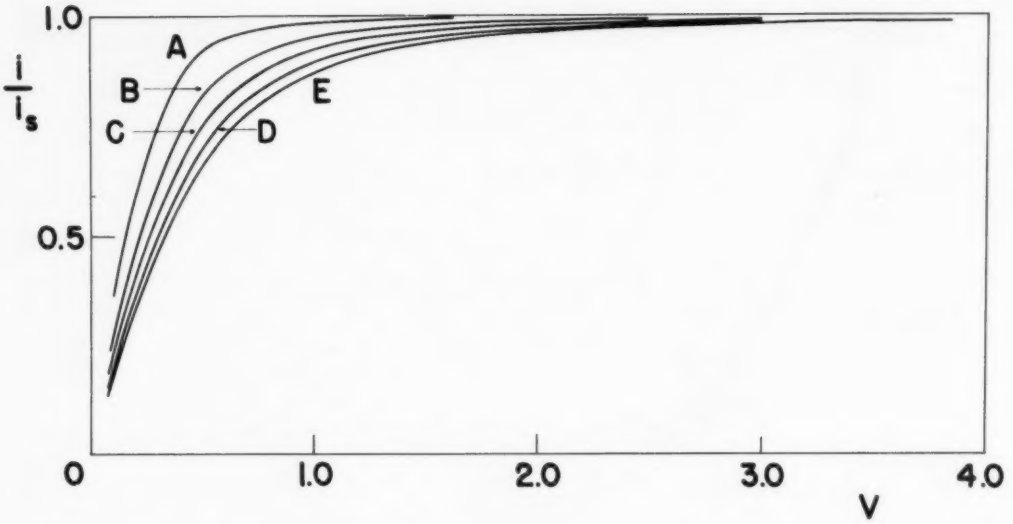

FIGURE 2. *Saturation curves assuming equal mobilities; computer solutions. Values of* α: *0.1 (A), 0.3(B), 0.5 (C), 0.7 (D), 0.9 (E).* ($\mu'_+ = \mu'_- = 1.89\times10^{-4} m^2/V\,sec$)

5. Charge Profiles

Once i is known we can perform a second integration of eqs (8) and (10), obtaining the functions $u(x)$, $E^*(x)$, and $V^*(x)$. Using eqs (6) one also obtains the variation of positive and negative space charge densities with x.

5.1. Equal Carrier Mobilities

Figure 3 shows the positive space charge density as a function of x for equal mobilities and $\alpha = 0.662$. The values of the negative charge density are symmetrical to the given curve with regard to $x = 1/2$. Different curves refer to different saturation ratios i/i_S. For low and high values of this ratio the charge density is a monotonously increasing function of x but in the intermediate region a maximum is found to occur. Physically this can be understood as follows: as one moves from anode to cathode the density of positive charge is expected to increase due to the cumulative flow of positive charge carriers moving out from the anode region. The velocity of the carriers is, however, modulated by the field. In the center the field is weakened and the velocity reduced. In the cathode region the field is increased and the carriers sped up. Flow lines are "spread out." With increasing field strength the latter effect predominates and the positive charge density falls off. "Carriers are sucked into the cathode." In the anode region the cumulative effect of carrier movement always predominates. Therefore, the curves have never more than one maximum. When approaching saturation the field becomes uniform and the effect disappears again.

This view is confirmed if one solves the basic equations for $\alpha = 0$, e.g., neglecting recombination; in this case one has always $i/i_S = 1$. Boag and Wilson obtained approximate analytical solution, assuming that the mobility of the negative carriers is very great compared to that of the positive carriers. A precise analytical solution can, however, be obtained also in the general case where the mobilities have arbitrary values or are equal. We shall omit here the

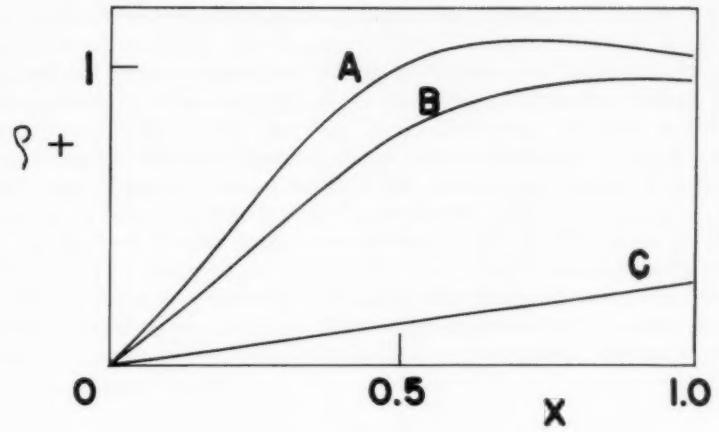

FIGURE 3. *Charge profiles–positive space charge density versus depth. Computer solution for equal mobilities and* $\alpha = 0.662$. *A)* $i/i_s = 0.6$, *B)* $i/i_s = 0.74$, *C)* $i/i_s = 0.99$ $(\mu'_+ = \mu'_- = 1.89 \times 10^{-4} m^2/V\,sec)$

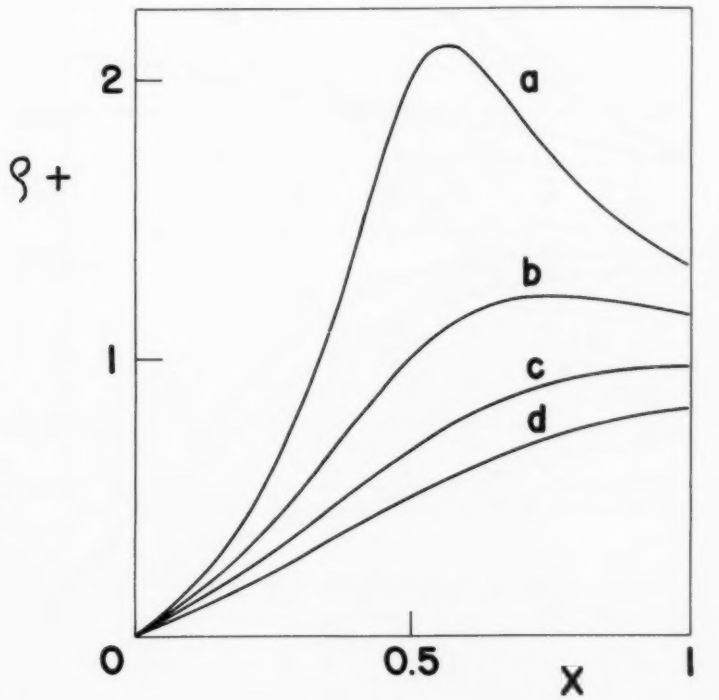

FIGURE 4. *Charge profiles without recombination for equal mobilities: analytic solution without recombination. a)* $V_0 = 0.45$, *b)* $V_0 = 0.64$, *c)* $V_0 = 0.85$, *d)* $V_0 = 1.08$ *(for an ionization chamber of 1 cm electrode distance and a dose rate of 1 R/sec,* $V_0 = 1$ *gives a voltage of 4.35 volt).*

calculus which is straightforward giving in figure 4 only the results obtained for the space charge density at different applied voltages. Comparison with the curves of figure 3 shows that with recombination the maxima are very much lower and occur nearer to the electrode. Thus recombination displaces the depth of the maxima toward the center of the chamber and reduces their amplitudes.

The effect is reflected in the behavior of the difference of space-charge densities $u = \rho_+ - \rho_-$. For low values of the saturation ratio, $u(x)$ increases nearly linearly with x. As i/i_S increases, ρ_+ increases in the cathode region and begins to be depleted in the anode region while the contrary happens to ρ_-. This is shown in figure 5, again for various values of the saturation ratio, including recombination.

Maxima of the space-charge density have been shown to occur for $\alpha = 0$ and $\alpha = 0.662$. The situation is different for much higher values of α. It can be shown f.i. that for $\alpha = 1$ maxima do not occur any more.

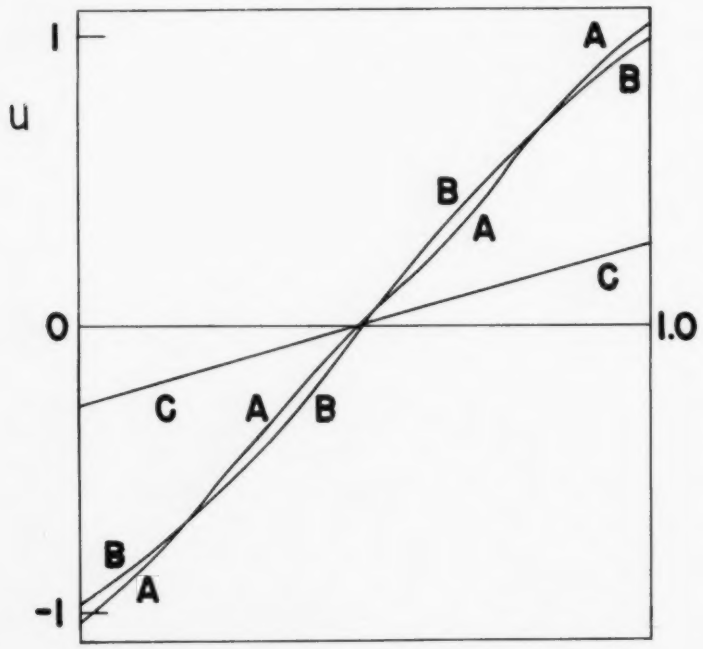

FIGURE 5. *Charge profiles–difference of space charge densities versus depth: computer solution for equal mobilities and* $\alpha = 0.662$. *A)*$i/i_s = 0.6$, *B)*$i/i_s = 0.75$, *C)*$i/i_s = 0.99$ $(\mu'_+ = \mu'_- = 1.89 \times 10^{-4} m^2/V\,sec)$.

5.2. Different Carrier Mobilities

We have extended the computation of charge densities so as to include also the case of different carrier mobilities, figure 6 gives results obtained for the mobility values and recombination coefficient of Scott and Greening [5] and the saturation ratios of $i/i_S = 0.68$, 0.77, and 0.98. Figure 6 shows that the curves for the less mobile positive carriers do not have maxima within the given interval of saturation ratios. The space charge density of the more mobile negative carriers does, however, exhibit a maximum for the lowest saturation ratio.

For equal mobilities, symmetry demands that the net charge density becomes zero at $x_0 = 1/2$. Figure 7 shows that for different mobilities the zero is slightly displaced toward the anode, but still is practically independent of the saturation ratio. Mie has discussed this point [1], page 884. His first approximation $x_0 = (\kappa - 1)/(\kappa + 1)$ is in good agreement with the computer results. Agreement becomes less satisfactory for his second approximation (eq 60), as can also be verified by inspection of figure 5, reference 4.

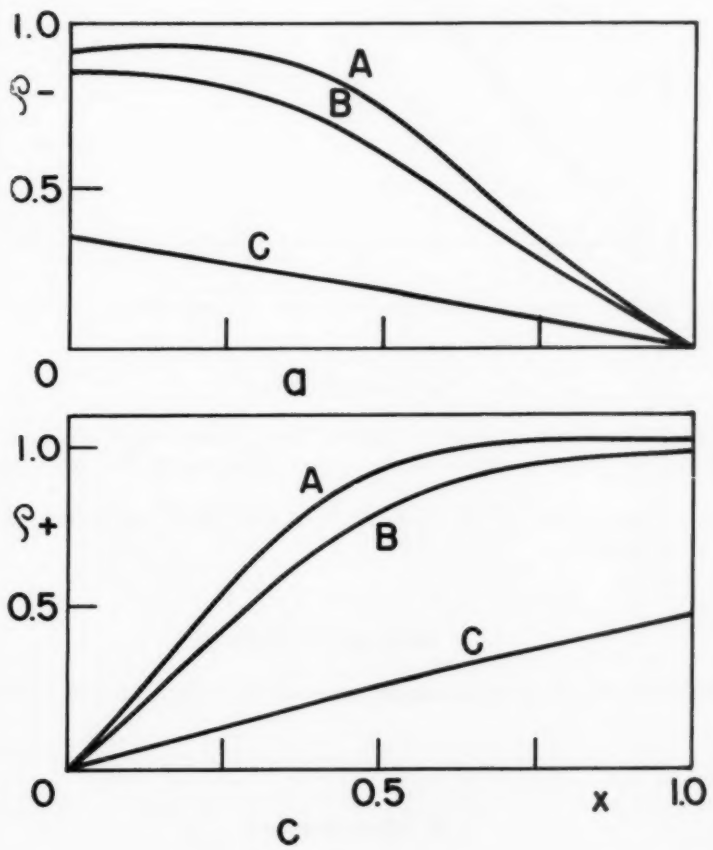

FIGURE 6. *Charge profiles—positive and negative charge densities versus depth. Computer solution for different mobilities* ($\mu'_+ = 1.59 \times 10^{-4} m^2/V$ *sec*, $\mu'_- = 2.21 \times 10^{-4} m^2/V$ *sec*) *and* $\alpha = 0.7999$. *A*) $i/i_s = 0.68$, *B*) $i/i_s = 0.77$, *C*) $i/i_s = 0.98$. *a*) positive charge density; *b*) negative charge density.

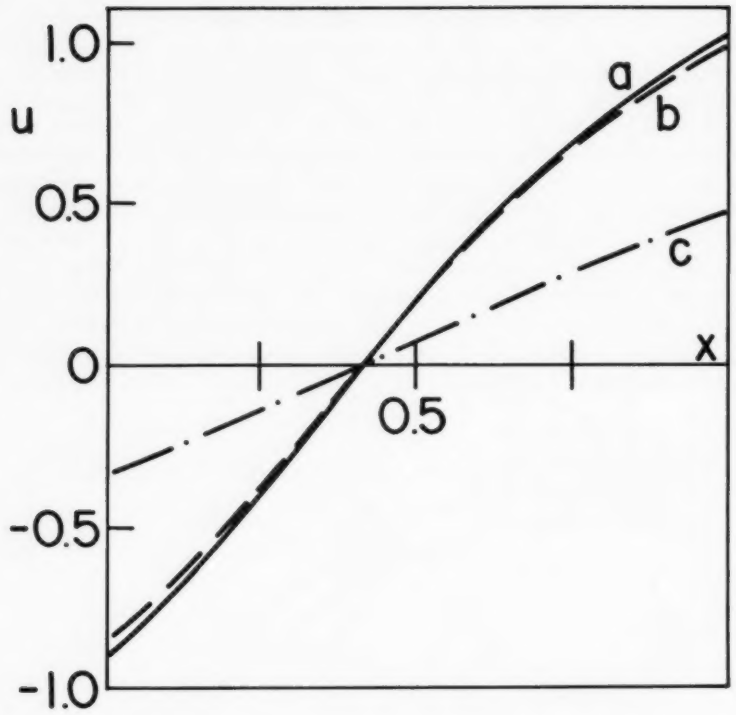

FIGURE 7. *Charge profiles–net space charge density versus depth. Computer solution–parameters as in figure 6.*

6. Conclusion

The present treatment shows that the positive and negative space-charge densities exhibit maxima which are due to the variation of the carrier velocities in the space-charge modulated electric field.

Calculations were carried out on a Hewlett Packard 9100 A calculator, using the Runge–Kutta method of integration.

7. Acknowledgment

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TABLE 1. *List of quantities and symbols*

Quantity	Symbol and dimension	Reduced (nondimensional) quantity
Electronic charge	$e = 1.602 \times 10^{-19}$ C	
Dielectric constant	$\epsilon = 8.85 \times 10^{-12}$ F/m	
Mobility of positive carriers	μ'_+ (m^2/V sec)	$\mu_+ = 1$
Mobility of negative carriers	μ'_- (m^2/V sec)	$\mu_- = \kappa$
Mobility ratio	κ	$\mu'_-/\mu'_+ = \kappa$
Recombination coefficient	α' (m^3/sec)	$\alpha = \alpha'\epsilon/e\mu_+$
Rate of ionization	q (C/m^3 sec)	
Length	x' (m)	$x = x'/d$
Electrode distance	d (m)	
Space charge density	ρ' (C/m^3)	$\rho = \rho'(\mu_+/\epsilon q)^{1/2}$
Difference of space charge densities		$v = \rho_+ - \rho_-$
Current density	i' (A/m^2)	$i = i'/qd$
Saturation current density	i'_s(A/m^2)	$i_s = i'_s/qd = 1$
Saturation ratio	i'/i'_s	$i = i'/i'_s$
Electric field strength	E' (V/m)	$E = (\epsilon\mu_+/q)^{1/2} E'/d$
Voltage	V' (V)	$V = (\epsilon\mu_+/q)^{1/2} V'/d^2$
Applied voltage	V'_0 (V)	$V_0 = (\epsilon\mu_+/q)^{1/2} V_0/d^2$
		$E^* = E/i$
		$V^* = V/i^2$

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Let G be a permutation group of order $|G|$ acting on a set of n elements. For $g \in G$, let $\theta(g)$ be the number of elements left fixed by g. Then, for $1 \leq r \leq n$,

$$\sum_{g \in G} (\theta(g))^r > X_r |G|$$

with equality if and only if G is r-fold transitive. Here, X_r is the r-th Bell number.

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